The status of order-preserving Martin's Conjecture

Benjamin Siskind[1]

(joint work with Patrick Lutz)

Leeds Logic Seminar
November 6, 2024

[1] This research was funded in whole or in part by the Austrian Science Fund (FWF) [10.55776/Y1498].

Outline

Outline

Background

There seems to be significant global structure to the functions on the Turing degrees which come up naturally in computability theory.

Martin's Conjecture is a precise way of stating that the global structure is really as it seems.

Roughly, the conjecture is that, under determinacy, the functions on the Turing degrees are minor variations of constant functions, the identity, and (transfinite) iterates of the Turing jump.

Background

Some basic definitions:

- For $x, y \in \mathbb{R}$, $x \leq_T y$ iff x is computable from y and $x \equiv_T y$ iff x is computable from y and vice-versa.
- A function $f : \mathbb{R} \to \mathbb{R}$ is *Turing-invariant* iff $x \equiv_T y \Rightarrow f(x) \equiv_T f(y)$.

A Turing-invariant function induces a function on the Turing degrees. Under AC, every function on the Turing degrees arises in this way. This will also be true in our contexts of interest even without AC.

Background

Some Turing-invariant functions:

- Constant functions,
- The identity function,
- The Turing jump: $x \mapsto x' =$ the Halting Problem relativized to x, a universal $\Sigma^0_1(x)$ subset of ω
- Finite iterates of the jump: $x \mapsto x'' = (x')'$, $x \mapsto x'''$, ...
- Transfinite iterates: $x \mapsto x^{(\alpha)}$ for any $\alpha < \omega_1$,
- $x \mapsto \mathcal{O}^x =$ Kleene's $\mathcal{O}$ relativized to x, a universal $\Pi^1_1(x)$ subset of ω,
- $x \mapsto x^{\#} =$ the theory of Gödel's $L[x]$ (in the language with symbols for x-indiscernibles)

Background

Anything else?

- We can make minor variations on the functions we've already seen, e.g. $F(x) = \begin{cases} 0 & \text{if } x \not\geq_T 0' \\ x' & \textit{otherwise} \end{cases}$.
- If we have access to some well-order $\prec$ of $\mathbb{R}$, we can make genuinely different functions, e.g.

$$F(x) = \begin{cases} 0 \text{ if } x \text{ is computable} \\ \text{the } \prec\text{-least } y \text{ such that } x \not\leq_T y \text{ and } y \not\leq_T x \text{ otherwise} \end{cases}$$

Background

This last example seems to make essential use of choice.
However, it is a theorem of ZF that there is no Borel well-order of the reals, so we can't get Borel functions in this way.

Assuming the existence of large cardinals, there are no reasonably definable well-orders of the reals; that is, no well-orders in $L(\mathbb{R})$, the minimal transitive model M of ZF containing the reals and the ordinals.

So maybe we can avoid this kind of example by restricting to Borel functions or functions in $L(\mathbb{R})$...

Background

Borel sets are so nice because they are *determined*: in every two player game with perfect information on the natural numbers with Borel pay-off set, one of the players has a winning strategy.

In the presence of large cardinals, every set of reals in $L(\mathbb{R})$ is also determined, i.e. $L(\mathbb{R}) \models \mathrm{AD}$, the Axiom of Determinacy.

Maybe the explanation of the apparent global structure of functions on the Turing degrees is that there are no pathological functions under AD...

Outline

Martin's Conjecture

- A *cone* is a set of the form $\{x \in \mathbb{R} \mid x \geq_T b\}$ for some real b.
- A set of reals is *Turing-invariant* iff it is closed under $\equiv_T$.

Theorem (Martin, '68)

Assume AD. *Any Turing-invariant set of reals either contains a cone or is disjoint from a cone.*

Definition

For $F, G : \mathbb{R} \to \mathbb{R}$ Turing-invariant, put $F \leq_M G$ iff $F(x) \leq_T G(x)$ on a cone of $x \in \mathbb{R}$ and $F \equiv_M G$ iff $F(x) \equiv_T G(x)$ on a cone of $x \in \mathbb{R}$.

Martin's Conjecture

We can now state Martin's Conjecture:

Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

1. For any Turing-invariant function $F : \mathbb{R} \to \mathbb{R}$, either there is a $c \in \mathbb{R}$ such that $F(x) \equiv_T c$ on a cone or $F(x) \geq_T x$ on a cone.
2. $\leq_M$ prewellorders the Turing-invariant functions F such that $F(x) \geq_T x$ on a cone and the successors in this prewellorder are given by composition with the Turing jump.

Both parts of Martin's Conjecture are still open, even for Borel functions, but there have been partial results.

Slaman and Steel showed both parts of Martin's Conjecture hold for the class of *uniformly* Turing-invariant functions, which is strong evidence for the truth of the conjecture.

Outline

Order-preserving Martin's Conjecture

In this talk, we'll consider Martin's Conjecture restricted to the class of *order-preserving* functions, which turns out to be more tractable.

Definition

A function $F : \mathbb{R} \to \mathbb{R}$ is *order-preserving* iff $x \leq_T y \Rightarrow F(x) \leq_T F(y)$.

Slaman and Steel also showed

Theorem (Slaman-Steel, '88)

Assume ZF. Part 2 of Martin's Conjecture holds for Borel order-preserving functions.

Order-preserving Martin's Conjecture

We showed

Theorem (Lutz-S., 2021)

Assume AD*. Part 1 of Martin's Conjecture holds for order-preserving functions.*

Combining this with the Slaman-Steel result, both parts of Martin's Conjecture hold for Borel order-preserving functions. Moreover, this is actually provable from just ZF.

So what remains of Martin's Conjecture for order-preserving functions is part 2 for order-preserving beyond the Borel ones.

Order-preserving Martin's Conjecture

Combining Slaman and Steel's work with a result of Woodin gives a bit more than part 2 of Martin's Conjecture for Borel order-preserving functions.

Theorem (Slaman-Steel-Woodin, '88)

Assume AD*. Let $F : \mathbb{R} \to \mathbb{R}$ be order-preserving such that $F(x) \geq_T x$ on a cone. Then either $F(x) \equiv_T x^{(\alpha)}$ on a cone for some $\alpha < \omega_1$ or $F(x) \geq_T \mathcal{O}^x$ on a cone.*

Recently, we showed

Theorem (Lutz-S.)

Assume AD*. Let $F : \mathbb{R} \to \mathbb{R}$ be order-preserving such that $F(x) \geq_T \mathcal{O}^x$ on a cone. Then either $F(x) \equiv_T (\mathcal{O}^x)^{(\sigma(\omega_1^x))}$ on a cone for some $\sigma : \omega_1 \to \omega_1$ or $F(x) \geq_T \mathcal{O}^{\mathcal{O}^x}$ on a cone.*

Outline

Mouse operators

I'll discuss this from an inner-model-theoretic perspective which we believe is important for trying to go further.

Jensen used his fine structure theory to identify canonical constructible reals, the so-called 'mastercodes' for levels of L.

Theorem (Jensen)

Let $\alpha < \omega_1^{L[x]}$ and suppose there is a $\Delta_{n+1}(J_\alpha[x])$ real which is not $\Delta_n(J_\alpha[x])$. Then there is one of maximum Turing degree, i.e. there is an $y \in \mathbb{R}$ which is $\Delta_{n+1}(J_\alpha[x])$ such that every z that is $\Delta_{n+1}(J_\alpha[x])$ is computable from y.

We call such a real y a $\Delta_{n+1}(J_\alpha[x])$ *mastercode*.

Mouse operators

Definition

A *baby mouse operator* is a function M with domain $\mathbb{R}$ such that for all $x \in \mathbb{R}$,

- $M(x) = J_\alpha[x]$ for some $\alpha < \omega_1^{L[x]}$ and
- for any $y \equiv_T x$, $M(y) = M(x)$.

A baby mouse operator M is *relevant* iff for all $x \in \mathbb{R}$ there is some $n \in \omega$ such that there is a $\Delta_{n+1}(M(x))$ real which is not $\Delta_n(M(x))$.

Given a relevant baby mouse operator M we let F_M denote any function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x)$ is a $\Delta_{n+1}(M(x))$ mastercode for n least such that there is a $\Delta_{n+1}(M(x))$ real which is not $\Delta_n(M(x))$.

Given a relevant baby mouse operator M, F_M is unique up to $\equiv_M$.

Mouse operators

Here are some examples of relevant baby mouse operators M and associated functions F_M.

- $x \mapsto J_1[x]$ and $x \mapsto x^{(\omega)}$,
- $x \mapsto J_{\omega_1^x}[x]$ and $x \mapsto \mathcal{O}^x$,
- $x \mapsto J_{\omega_1^x+1}[x]$ and $x \mapsto (\mathcal{O}^x)^{(\omega)}$,
- $x \mapsto J_{\omega_2^x}[x]$ and $x \mapsto \mathcal{O}^{\mathcal{O}^x}$.

By work of Rudominer, the notion of mastercodes can be extended beyond levels of $L[x]$, which leads us to relevant mouse operators beyond the baby ones. Essentially, these are operators sending x to a sound mouse over x that defines a new real.

Here are examples of this more general notion of a relevant mouse operator and associated mastercode function.

- $x \mapsto$ the premouse $x^\#$ and $x \mapsto$ the real $x^\#$,
- $x \mapsto M_1^\#(x)$ and $x \mapsto$ the real $M_1^\#$.

Mouse operators

We can identify sufficient criteria for establishing part 2 of Martin's Conjecture for order-preserving functions up to F_M for M a relevant mouse operator, based on Slaman and Steel's work.

Moreover, these criteria are just generalizations of well-known theorems from computability theory: the Friedberg jump inversion theorem and the Posner-Robinson theorem.

Mouse operators

Theorem (Friedberg, '57)

Let $x, y \in \mathbb{R}$ and suppose $y \geq_T x'$. Then there is a $z \geq_T x$ such that $y \equiv_T z'$.

Definition

A relevant mouse operator M has the *Friedberg property* iff on a cone of x, for all $y \geq_T F_M(x)$ there is a $z \geq_T x$ such that $y \equiv_T F_M(z)$.

Friedberg's proof provides a general method of establishing the Friedberg property.

Fact (Friedberg, essentially)

Suppose that on a cone of x, $M(x \oplus y) = M(x)[y]$ whenever y is a sufficiently Cohen generic real. Then M has the Friedberg property.

Mouse operators

Theorem (Posner-Robinson, '81)

Suppose $x, y \in \mathbb{R}$ are such that $y \not\leq_T x$. Then there is $z \geq_T x$ such that $y \oplus z \geq_T z'$.

Definition

A relevant mouse operator M has the *Posner-Robinson property* iff on a cone of x, for all $y \geq_T x$ such that $y \notin M(x)$, there is a $z \geq_T x$ such that $y \oplus z \geq_T F_M(z)$.

The original proof of Posner-Robinson does not generalize but Kumabe and Slaman found a forcing proof which does generalize.

Fact (Kumabe-Slaman, essentially)

Suppose that on a cone of x, $M(x \oplus y) = M(x)[y]$ whenever y is a sufficiently Kumabe-Slaman generic real. Then M has the Posner-Robinson property.

Mouse operators

If we could show all relevant mouse operators have both properties, we then would get that order-preserving Martin's Conjecture holds in models of AD^+, a technical strengthening of AD, which additionally satisfy *Mouse Capturing* (MC): if x is ordinal definable from y, then there is a sound mouse M over y such that $x \in M$.

Theorem

Assume $\mathsf{AD}^+ + \mathsf{MC}$. *Suppose every relevant mouse operator M has the Friedberg property and the Posner-Robinson property. Then part 2 of Martin's Conjecture holds for all order-preserving functions.*

By results of Woodin and Steel, $L(\mathbb{R})$ actually satisfies $\mathsf{AD}^+ + \mathsf{MC}$ under large cardinals (or even just when $L(\mathbb{R}) \models \mathsf{AD}$).

Mouse operators

The previous result actually holds locally; for example we also get the following.

Theorem

Assume AD. *Suppose every relevant baby mouse operator M has the Friedberg property and the Posner-Robinson property. Then part 2 of Martin's Conjecture holds for all order-preserving functions F such that $F(x) \in L[x]$ on a cone.*

We can prove that many natural mouse operators have both properties, but there are baby mouse operators very low down for which we can't establish either property.

Outline

Main theorem

Definition

A function $g : \mathbb{R} \to \omega_1$ is *Turing-invariant* iff $x \equiv_T y \Rightarrow g(x) = g(y)$. For $g, h : \mathbb{R} \to \omega_1$ Turing-invariant, put $g \leq_M h$ iff $g(x) \leq_M h(x)$ on a cone.

Fact

Assume AD. $\leq_M$ *prewellorders the Turing-invariant functions* $g : \mathbb{R} \to \omega_1$.

Definition

A Turing-invariant function $g : \mathbb{R} \to \omega_1$ is *regular* iff there is no function H from $\mathbb{R}$ to countable sets of ordinals such that

1. if $x \equiv_T y$, then $H(x) = H(y)$,
2. $H(x)$ is a cofinal subset of $g(x)$ of order-type $< g(x)$.

For $g : \mathbb{R} \to \omega_1$ Turing-invariant, let $M_g(x) = J_{g(x)}[x]$.

Main theorem

Theorem (Lutz-S.)

Assume AD. *Let* $g : \mathbb{R} \to \omega_1$ *be a* $<_M$*-minimal Turing-invariant function such that* g *is regular and for all* x*,* $g(x)$ *is a limit of* x*-admissibles. Then for any Turing-invariant* $h : \mathbb{R} \to \omega_1$ *such that* $h <_M g$*,* M_h *has the Friedberg property and the Posner-Robinson property.*

As a corollary we get

Theorem (Lutz-S.)

Assume AD. *Let* $g : \mathbb{R} \to \omega_1$ *be a* $<_M$*-minimal Turing-invariant function such that* g *is regular and for all* x*,* $g(x)$ *is a limit of* x*-admissibles. Then part 2 of Martin's Conjecture holds for order-preserving functions* F *such that* $F(x) \in M_g(x)$ *on a cone.*

In particular, this gives that part 2 of Martin's Conjecture holds for order-preserving up to many transfinite iterates of the hyperjump.

Main theorem

Key question:

What is g?
($g : \mathbb{R} \to \omega_1$ is a $<_M$-minimal Turing-invariant function such that g is regular and for all x, $g(x)$ is a limit of x-admissibles.)

The natural candidate is the function taking x to the least x-recursively inaccessble ordinal, i.e. the least x-admissible limit of x-admissibles.

We can actually show that $g(x)$ is *at most* the least x-recursively inaccessble ordinal on a cone. If we could show they are equivalent on a cone, we could go further.

The End

Thanks!