Convolution semigroup on Keisler measures

Artem Chernikov
(joint with Kyle Gannon and Krzysztof Krupiński)

University of Maryland, College Park

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Classical (locally-)compact case

- Let G be a locally compact topological group.
- Then the space $\mathcal{M}(G)$ of regular Borel probability measures on G is equipped with the *convolution product*:

$$\mu * \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y) d\mu(x) d\nu(y)$$

  for a Borel set $A \subseteq G$.
- A measure μ is *idempotent* if $\mu * \mu = \mu$.
- A classical line of work in progressively broader contexts [Kawada,Itô'40], [Wendel'54], [Rudin'59], [Glicksberg'59], [Cohen'60] culminates in:

Fact (Pym'62)

Let G be a locally compact group and $\mu \in \mathcal{M}(G)$. Then the following are equivalent:

1. *μ is idempotent;*
2. *the support* $\mathrm{supp}(\mu)$ *of μ is a compact subgroup of G and $\mu|_{\mathrm{supp}(\mu)}$ is the normalized Haar measure on* $\mathrm{supp}(\mu)$*.*

Idempotent types in stable groups

- Generalizing a classical fact about idempotent types in stable groups [Newelski]:

Fact (C., Gannon)

Let G be a (type-)definable group in a stable structure M, $\mathcal{U} \succ M$ a saturated elementary extension, and $\mu \in \mathfrak{M}_{G,M}(\mathcal{U})$ a global Keisler measure. Then μ is idempotent if and only if μ is the unique left-invariant (and the unique right-invariant) measure on a type-definable subgroup of $G(\mathcal{U})$ (namely, the left-/right-stabilizer of μ).

- The following groups are stable: abelian, free, algebraic over $\mathbb{C}$ (e.g. $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$, abelian varieties), etc.
- This suggests a remarkable analogy between the topological and definable settings, even though they are proved using rather different methods.
- We are interested in extending this beyond stable structures, especially to generically stable measures in NIP groups — non-trivial already for types instead of general measures.

Invariant types and Morley sequences

Given a global type $p \in S_x(\mathcal{U})$ (automorphism-)invariant over a small set $A \subseteq \mathcal{U}$ and $\mathcal{U}' \succ \mathcal{U}$ a bigger monster model with respect to $\mathcal{U}$, we let $p|_{\mathcal{U}'}$ be the unique extension of p to a type in $S_x(\mathcal{U}')$ which is invariant over A. Given another A-invariant type $q \in S_y(\mathcal{U})$, we define the A-invariant type $p \otimes q \in S_{xy}(\mathcal{U})$ via $p \otimes q := \mathrm{tp}(ab/\mathcal{U})$ for some/any a, b in $\mathcal{U}'$ such that $b \models q$ and $a \models p|_{\mathcal{U}b}$. Given an arbitrary linear order $(I, <)$, a sequence $\bar{a} = (a_i : i \in I)$ in $\mathcal{U}$ is a *Morley sequence* in p over A if $a_i \models p|_{Aa_{<i}}$ for all $i \in I$. Then the sequence $\bar{a}$ is indiscernible over A, and for any other Morley sequence $\bar{a}' = (a'_i : i \in I)$ in p over A we have $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{a}'/A)$. We can then define a global A-invariant type $p^{(I)}((x_i : i \in I)) \in S_{\bar{x}}(\mathcal{U})$ as $\bigcup\{\mathrm{tp}(\bar{a}/B) : A \subseteq B \subseteq \mathcal{U} \text{ small}, \bar{a} = (a_i : i \in I) \text{ a Morley sequence in } p \text{ over } B\}$.

Generically stable types, 1

- *Stable structures* are viewed as a model theoretic paradise, and many tools for analyzing types and models are available.
- When considering larger classes of structures, for example NIP, *generically stable types*, when available, play an important role both in their model theoretic analysis and applications (elimination of imaginaries in ACVF; model theoretic counterpart of Berkovich spaces; etc.)
- Many equivalent characterizations under NIP, working in an arbitrary theory use the strongest one [Pillay, Tanovic]:

Definition

A global type $p \in S_x(\mathcal{U})$ is *generically stable* if it is A-invariant for some small $A \subset \mathcal{U}$, and for any ordinal α (or just for $\alpha = \omega + \omega$), $(a_i : i \in \alpha)$ a Morley sequence in p over A and formula $\varphi(x) \in \mathcal{L}(\mathcal{U})$, the set $\{i \in \alpha : \models \varphi(a_i)\}$ is either finite or co-finite.

Generically stable types, 2

Fact

Let $p \in S_x(\mathcal{U})$ be generically stable, invariant over $A \subseteq \mathcal{U}$. Then:

1. *Every realization of $p^{(\omega)}|_A$ is a totally indiscernible sequence over A.*
2. *The type p is the unique global non-forking extension of $p|_A$.*
3. *For any $a \models p|_A$ and b in $\mathcal{U}$ such that $\mathrm{tp}(b/A)$ does not fork over A, we have $a \mathop{\smile\!\!\!\!\!\!|}_A b \iff b \mathop{\smile\!\!\!\!\!\!|}_A a$ (this holds for any b when A is an extension base, e.g. when $A \prec \mathcal{U}$).*
4. *In particular, if $a, b \models p|_A$, then $a \mathop{\smile\!\!\!\!\!\!|}_A b \iff (a, b) \models p^{(2)}|_A \iff (b, a) \models p^{(2)}|_A$.*
5. *If A is an extension base, $(a_i)_{i<\omega} \models p^{(\omega)}|_A$ and $\varphi(x, a_0)$ (where $\varphi(x, y) \in \mathcal{L}(A)$) forks/divides over A, then $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent.*
6. *Let $a \models p|_A$ and let b, c be arbitrary small tuples in $\mathcal{U}$. If $a \mathop{\smile\!\!\!\!\!\!|}_A b$ and $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} c$, then $a \mathop{\smile\!\!\!\!\!\!|}_A bc$;*
7. *p is definable over A.*

Generically stable groups

- Let $G = G(x)$ be an $\emptyset$-type-definable group. For $A \subseteq \mathcal{U}$, $S_G(A)$ denotes the set of types $p \in S(A)$ *concentrated on* G, i.e. such that $p(x) \vdash G(x)$

Definition (Pillay, Tanovic)

A type-definable group $G(x)$ is *generically stable* if there is a generically stable $p \in S_G(\mathcal{U})$ which is left $G(\mathcal{U})$-invariant (we might use "$G(\mathcal{U})$-invariant" and "G-invariant" interchangeably when talking about global types).

Fact

Suppose that G is a generically stable type-definable group in an arbitrary theory, witnessed by a generically stable type $p \in S_G(\mathcal{U})$. Then we have:

1. *p is the unique left $G(\mathcal{U})$-invariant and also the unique right $G(\mathcal{U})$-invariant type;*
2. *$p = p^{-1}$ (where $p^{-1} := \mathrm{tp}(g^{-1}/\mathcal{U})$ for some/any $g \models p$ in a bigger monster model $\mathcal{U}' \succ \mathcal{U}$).*

Idempotent generically stable types

- Let $p \in S_G(\mathcal{U})$ be a generically stable type over M.
- The left stabilizer $\mathrm{Stab}(p)$ of p is an intersection of relatively M-definable subgroups of G; in particular, it is M-type-definable.
- Given $p, q \in S_G(\mathcal{U})$ global M-invariant types, we define $p * q \in S_G(\mathcal{U})$ via $p * q(\varphi(x)) := p_x \otimes q_y(\varphi(x \cdot y))$ for all $\varphi(x) \in \mathcal{L}(U)$. Together with this operation, the set of all global M-invariant types in $S_G(\mathcal{U})$ forms a left-continuous semigroup.
- We say that an invariant type $p \in S_G(\mathcal{U})$ is *idempotent* if $p * p = p$.

Example

Let G' be an arbitrary type-definable subgroup of G which is generically stable, witnessed by a generically stable left or right G'-invariant type $p \in S_{G'}(\mathcal{U})$. Then p is obviously idempotent.

- Our central question in the case of types is whether this is the only source of generically stable idempotent types.

Stabilizers

- We let $H_\ell := \mathrm{Stab}_\ell(p)$ and $H_\mathrm{r} := \mathrm{Stab}_\mathrm{r}(p)$ be the left and the right stabilizer of p, respectively. Write H for either H_ℓ or H_r.

Proposition

Assume p is generically stable and $p \in S_H(\mathcal{U})$. Then:

1. *H is a generically stable group, witnessed by p (hence p is both the unique left-invariant and the unique right-invariant type of H);*
2. *H is the smallest among all type-definable subgroups H' of G with $p \in S_{H'}(\mathcal{U})$;*
3. *H is both the left and the right stabilizer of p in G.*

Generic transitivity

Proposition

The following conditions are equivalent for a generically stable p:

1. $p \in S_{H_\ell}(\mathcal{U})$;
2. $a \in \mathrm{Stab}_\ell(p')$ *(where $p' := p|_{\mathcal{U}'}$ is the unique extension of p to a bigger monster model $\mathcal{U}'$ invariant over M);*
3. *for any/some $(a_0, a_1) \models p^{(2)}$, $(a_0 \cdot a_1, a_0) \models p^{(2)}$;*
4. *Same with "right" instead of "left": $p \in S_{H_r}(\mathcal{U})$; $a \in \mathrm{Stab}_r(p')$; for any/some $(a_0, a_1) \models p^{(2)}$, $(a_1 \cdot a_0, a_0) \models p^{(2)}$.*

Definition

We will say that a generically stable type $p \in S_G(\mathcal{U})$ is *generically transitive* if it satisfies any of these equivalent conditions.

Problem

Assume that p is generically stable and idempotent. Is it then generically transitive?

Main theorem for types

Theorem

Assume $p \in S_G(\mathcal{U})$ is generically stable and idempotent, and one of the following holds:

1. *p is stable and M is arbitrary;*
2. *G is abelian and M is arbitrary;*
3. *G is arbitrary and M is inp-minimal;*
4. *G is arbitrary and M is rosy (e.g. if $\mathrm{Th}(M)$ is simple).*

Then p is generically transitive, hence it is the unique left-/right-invariant type on a type-definable subgroup of $G(\mathcal{U})$ (namely, the left-/right-stabilizer of p).

- Remains open for general NIP groups.
- (2) and (3) rely on weight arguments, while (1), (4) rely on stratified rank arguments.

Generic transitivity = stable group theory localized at p

- Generic transitivity is a sufficient and necessary condition for developing some crucial results of stable group theory localizing on a generically stable type.
- **Theorem 1.** In an arbitrary theory, there is an analog of the stratified rank in stable theories restricting to subsets of $G(\mathcal{U})$ definable using parameters from a Morley sequence in a generically stable type p. This rank is finite, and it is is left invariant (under multiplication by realizations of p) iff p is generically transitive.
- **Theorem 2.** (Adapting Hrushovski) If G is type-definable and $p \in S_G(\mathcal{U})$ is generically stable, idempotent and generically transitive, then its stabilizer is an intersection of M-definable groups.
- **Theorem 3.** A chain condition for groups type-definable using parameters from a Morley sequence of a generically stable type p holds, implying that there is a smallest group of this form — and it is equal to the stabilizer of p when p is generically transitive.

Keisler measures

- A *Keisler measure* μ in variables x over $A \subseteq \mathcal{U}$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of A-definable subsets of $\mathcal{U}_x$.
- $\mathfrak{M}_x(A)$ denotes the set of all Keisler measures in x over A.
- Then $\mathfrak{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

  with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A), r_i, s_i \in [0,1]$ for $i < n$.
- Identifying p with the Dirac measure δ_p, $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathfrak{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$.

Product and Morley sequences of Keisler measures

Definition

Let $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$ and suppose that μ is *Borel-definable*. Their Morley product $\mu \otimes \nu$ is the unique measure in $\mathfrak{M}_{xy}(\mathcal{U})$ such that for any $\varphi(x, y) \in \mathcal{L}_{xy}(\mathcal{U})$, we have

$$(\mu \otimes \nu)(\varphi(x, y)) = \int_{S_y(A)} F^{\varphi}_{\mu,A} d(\widehat{\nu|_A}), \text{ where:}$$

1. μ is A-invariant and A contains all the parameters from φ,
2. $F^{\varphi}_{\mu,A} : S_y(A) \to [0, 1]$ is defined by $F^{\varphi}_{\mu,A}(q) = \mu(\varphi(x, b))$ for some (equivalently, any) $b \models q$ in $\mathcal{U}$,
3. $\widehat{\nu|_A}$ is the unique regular Borel probability measure on $S_x(A)$ corresponding to the Keisler measure $\nu|_A$.

- We define $\mu^{(1)} := \mu(x_1)$,
  $\mu^{(n+1)}(x_1, \dots, x_{n+1}) := \mu(x_{n+1}) \otimes \mu^{(n)}(x_1, \dots, x_n)$, and
  $\mu^{(\omega)} = \bigcup_{n<\omega} \mu^{(n)}(x_1, \dots, x_n)$.

Generically stable measures, in arbitrary theories

- As for types [Pillay, Tanovic], in order to define generic stability for measures in arbitrary theories we want to take the strongest of the equivalent characterization under NIP.

Definition

[Hrushovski, Pillay, Simon] Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$ a small model. A Borel-definable measure μ is *fim* (a *frequency interpretation measure*) over M if μ is M-invariant and for any $\mathcal{L}$-formula $\varphi(x, y)$ there exists a sequence of formulas $(\theta_n(x_1, \ldots, x_n))_{1 \leq n < \omega}$ in $\mathcal{L}(M)$ such that:

1. for any $\varepsilon > 0$, there exists some $n_\varepsilon \in \omega$ satisfying: for any $k \geq n_\varepsilon$, if $\mathcal{U} \models \theta_k(\bar{a})$ then

$$\sup_{b \in \mathcal{U}^y} |\operatorname{Av}(\bar{a})(\varphi(x, b)) - \mu(\varphi(x, b))| < \varepsilon;$$

2. $\lim_{n \to \infty} \mu^{(n)}(\theta_n(\bar{x})) = 1$.

We say that μ is *fim* if μ is *fim* over some small $M \prec \mathcal{U}$.

Analog for compact groups: fim groups

Definition

An ($\emptyset$-)type-definable group $G(x)$ is *fim* if there exists a right G-invariant fim measure $\mu \in \mathfrak{M}_G(\mathcal{U})$ (where $\mathfrak{M}_G(\mathcal{U})$ is the space of measures supported on G), i.e. $\mu \cdot g = \mu$ for all $g \in G(\mathcal{U})$.

- A simultaneous generalization of [Pillay, Tanovic] for types in arbitrary theories, and of the previously known case for measures under the NIP assumption [Hrushovski, Pillay, Simon] (so an NIP group is fim iff it is *fsg*; fsg groups in *o*-minimal structures are precisely the definably compact groups):

Theorem

Suppose that $G(x)$ is a $\emptyset$-type-definable fim group, witnessed by μ. Then we have:

1. $\mu = \mu^{-1}$ *(where $\mu^{-1}(\varphi(x)) := \mu(\varphi(x^{-1}))$;*
2. μ *is left G-invariant;*
3. μ *is the unique left G-invariant measure in $\mathfrak{M}_G(\mathcal{U})$;*
4. μ *is the unique right G-invariant measure in $\mathfrak{M}_G(\mathcal{U})$.*

Convolution product of Keisler measures

Definition

Suppose that $\mu \in \mathfrak{M}_G(\mathcal{U})$ is Borel-definable. Then for any measure $\nu \in \mathfrak{M}_G(\mathcal{U})$, the (definable) *convolution of* μ *and* ν, denoted $\mu * \nu$, is the unique measure in $\mathfrak{M}_G(\mathcal{U})$ such that for any formula $\varphi(x) \in \mathcal{L}(\mathcal{U})$,

$$(\mu * \nu)(\varphi(x)) = (\mu \otimes \nu)(\varphi(x \cdot y)).$$

We say that μ is *idempotent* if $\mu * \mu = \mu$.

- When T is NIP, it is enough to assume that μ is automorphism-invariant, and $*$ is left-continuous.

Idempotent fim measures and generic transitivity

- Let $G(x)$ be a $\emptyset$-type-definable group. For $\mu \in \mathfrak{M}_G(\mathcal{U})$, we let $\mathrm{Stab}(\mu) := \{g \in G(\mathcal{U}) : \mu \cdot g = g\}$ be the right-stabilizer of μ.
- When $\mu \in \mathfrak{M}_G(\mathcal{U})$ is a measure definable over $M \prec \mathcal{U}$, then $\mathrm{Stab}(\mu)$ is an M-type-definable subgroup of $G(\mathcal{U})$.
- We let $H := \mathrm{Stab}(\mu)$ and $f : (\mathcal{U}^x)^2 \to (\mathcal{U}^x)^2$ be the ($\emptyset$-definable) map $f(x_1, x_0) = (x_1 \cdot x_0, x_0)$.

Proposition

Let $\mu \in \mathfrak{M}_G(\mathcal{U})$ be an idempotent fim measure. Then the following are equivalent:

1. $\mu \in \mathfrak{M}_H(\mathcal{U})$;
2. $\mu^{(2)} = f_* \left(\mu^{(2)}\right)$ *(i.e., the push-forward of* $\mu^{(2)}$ *under* f*);*
3. $\mu \otimes p = f_*(\mu \otimes p)$ *for every* $p \in S(\mu)$.

Definition

We say that an idempotent *fim* measure $\mu \in \mathfrak{M}_G(\mathcal{U})$ is *generically transitive* if it satisfies any of these equivalent conditions.

Idempotent fim measures and generic transitivity

Proposition

Assume μ is fim and $\mu \in \mathfrak{M}_H(\mathcal{U})$, where H is either left or right stabilizer of μ. Then:

1. *H is a fim group, hence μ is both the unique left-invariant and the unique right-invariant measure supported on H;*
2. *H is the smallest among all type-definable subgroups H' of G with $\mu \in \mathfrak{M}_{H'}(\mathcal{U})$;*
3. *H is both the left and the right stabilizer of μ in G.*

Example

If G' is a *fim* type-definable subgroup of G, witnessed by a G'-invariant *fim* measure $\mu \in \mathfrak{M}_{G'}(\mathcal{U})$, then μ is obviously idempotent and generically transitive.

Problem

Assume that $\mu \in \mathfrak{M}_G(\mathcal{U})$ is fim and idempotent. Is it true that then μ is generically transitive? Assuming T is NIP?

Idempotent fim measures and generic transitivity

Theorem

Assume that $G(x)$ is an abelian type-definable group and $\mu \in \mathfrak{M}_G(\mathcal{U})$ is fim and idempotent. Then μ is generically transitive.

- In particular, if T is NIP and G is abelian, there is a one-to-one correspondence between generically stable idempotent measures and type-definable fsg subgroups of G.
- Our proof generalizes the bounded local weight argument from the case of types in a purely measure theoretic context, using some theory of fim groups, arguments with push-forwards and the following general result about fim measures:

Generically stable measures over "random" parameters

Theorem

Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ *be fim over* M, $\nu \in \mathfrak{M}_y(\mathcal{U})$, $\varphi(x, y, z) \in \mathcal{L}_{xyz}$, $b \in \mathcal{U}^z$, *and* $\mathbf{x} = (x_i)_{i \in \omega}$. *Suppose that* $\lambda \in \mathfrak{M}_{\mathbf{x}y}(\mathcal{U})$ *is arbitrary such that* $\lambda|_{\mathbf{x},M} = \mu^{(\omega)}$ *and* $\lambda|_y = \nu$. *Then*

$$\lim_{i \to \infty} \lambda(\varphi(x_i, y, b)) = \mu \otimes \nu(\varphi(x, y, b)).$$

Moreover, for every $\varepsilon > 0$ *there exists* $n = n(\mu, \varphi, \varepsilon) \in \mathbb{N}$ *so that for any* ν, λ, b *as above, we have*
$\lambda(\varphi(x_i, y, b)) \approx^{\varepsilon} \mu \otimes \nu(\varphi(x, y, b))$ *for all but* n *many* $i \in \mathbb{N}$.

- Generalizes the usual characterization of generically stable measures in NIP (when ν is a type).
- Our proof relies on the use of Keisler randomization in continuous logic [Ben Yaacov, Keisler] and the correspondence between measures and their properties in T and types (in the sense of continuous logic) in its randomization T^R [Ben Yaacov] (studied further [Conant, Gannon, Hanson]).

Thank you!

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- "Definable convolution and idempotent Keisler measures II", Artem Chernikov, Kyle Gannon, Model Theory 2-2 (2023), 185-232
- "Definable convolution and idempotent Keisler measures III. Generic stability, generic transitivity, and revised Newelski's conjecture" Artem Chernikov, Kyle Gannon and Krzysztof Krupiński, arXiv:2406.00912