Pseudo-finite semigroups and diameter

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The idea

A semigroup S is said to be **right pseudo-finite** if $S \times S$ can be generated as a right congruence by a finite set $U \subseteq S \times S$, and there is a bound on the length of derivations for an arbitrary pair $(s, t) \in S \times S$ as a consequence of those in U.

If S is right pseudo-finite then the **right diameter** is the minimal bound over all finite sets of generators U.

Outline of talk

- Semigroups, acts and right congruences
- Pseudo-finite semigroups - background and motivation
- Pseudo-finite semigroups - minimal ideals
- Diameter
- Where to now?

Semigroups and monoids

A **semigroup** S is a non-empty set together with an associative binary operation.
A **monoid** is a semigroup with an identity 1.

Examples of semigroups

- Groups;
- Multiplicative semigroups of rings;
- Semigroups of transformations or relations, e.g.
- Full transformation monoid $\mathcal{T}_X$ on a set X;
- Endomorphism monoids of algebraic structures;
- Semigroups that are unions of groups a.k.a. completely regular semigroups;
- Bands;
- Finite semigroups.

Semigroups, acts and right congruences
What makes semigroups so interesting?

Semigroups are the first and most natural class of algebras we meet for which we need the tools of universal algebra.

A homomorphic image of a monoid is not determined by the class of the identity.
We need to consider **congruences** per se.
A cyclic right action of a monoid is not determined by the class of the identity.
We need to consider **right congruences** per se.

• A **right ideal** of S is a non-empty subset I such that $IS \subseteq I$; dually left

• An **ideal** of S is a subset I that is both a left and a right ideal.

• An ideal gives a congruences and a homomorphic image, but not vice versa.

S-act

A set A is a (right) S-act if there exists a map $A \times S \to A$ where $(a, s) \mapsto as$ such that for all $a \in A$ and $s, t \in S$ we have

$$a1 = a \text{ and } a(st) = (as)t.$$

Right ideals are right S-acts, so S is a right S-act.

A cyclic action is not determined by a right ideal. We need to consider **right congruences** per se.

Right congruences

A **right congruence** ρ on S is an equivalence relation ρ such that for every $a, b \in S$ and $c \in S$ we have

$$a \,\rho\, b \Rightarrow ac \,\rho\, bc.$$

A relation ρ on S is a subset of $S \times S$; we pass without mention between $a \,\rho\, b$ and $(a, b) \in \rho$.

- Clearly $\omega_r : S \times S$ and $\iota_S = \{(a, a) : a \in S\}$ are right congruences.
- Right congruences on a monoid S corresponds to a cyclic right action.
- Right congruences form a complete sublattice of the lattice of equivalence relations on S.
- A monoid S is **right noetherian** if every right congruence is finitely generated.
- How do we generate right congruences?

If $U \subseteq S \times S$, then there is a smallest right congruence $\langle U \rangle$ containing U, the **right congruence generated by** U.

Explicit form of $\langle U \rangle$

We have $a \langle U \rangle b$ if and only if $a = b$ or there exists a sequence

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \cdots,\ d_n t_n = b,$$

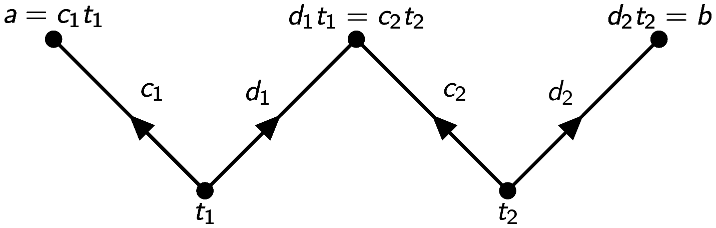
where $(c_i, d_i) \in U \cup U^{-1}$ and $t_i \in S$.
A sequence as above is a U-**sequence of length** n**.**

Pseudo-finiteness and diameter

A monoid S is **pseudo-finite** if $\omega_r := S \times S$ is finitely generated as a right congruence and there is an upper bound on the length n of the derivations required.
If S is pseudo-finite, the smallest upper bound (over all finite generating sets) is the **diameter** of S.

The notion of pseudo-finiteness for S-acts and for semigroups is analogous.

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ d_2 t_2 = b$$



Pseudo-finite semigroups
Background and motivation

- Pseudo-finite monoids were introduced by Dales and White (2017) to understand the relation between maximal ideals in semigroup algebras being finitely generated, and the algebra itself being finitely generated.
- Kobayashi (2007) showed ω_r being finitely generated is equivalent to S being right FP_1.
- Right (actually, left) FP_1 monoids are investigated in Kobayashi (2007), Pride & Gray (2011) and G., Quinn-Gregson, Zenab & Yang (2019). In the latter paper we also considered the case of pseudo-finiteness.

Pseudo-finite semigroups
Background and motivation

- If S has finitely generated right diagonal act $S \times S$, then S is pseudo-finite.
- If S is right noetherian then certainly ω_r is finitely generated; if S is pseudo-finite, then ω_r is finitely generated. Otherwise, the three conditions are independent.
- Pseudo-finiteness is a finitary condition, in that clearly all finite monoids are pseudo-finite (take $U = S \times S$).
- The use of finitary conditions have become embedded in standard algebraic practice over the last century. Where does pseudo-finiteness fit?

Pseudo-finite semigroups and minimal ideals
A few observations

A finite monoid S is pseudo-finite: take $U = S \times S$.

A finite monoid has a minimal ideal!!

Any monoid S with zero is pseudo-finite.

Take $U = \{(1, 0)\}$. Then for any $a, b \in S$ we have

$$a = 1a,\ 0a = 0b,\ 1b = b.$$

Certainly a monoid with zero has a minimal ideal.

Fact

A group G is such that ω_r is finitely generated if and only if G is a finitely generated group.
A group G is pseudo-finite if and only if it is finite.

Any group has a minimal ideal.

Pseudo-finite semigroups and minimal ideals

Dales and White conjectured that any pseudo-finite monoid is exactly $M \times F$ where M has a zero and F is finite.

Their conjecture was incorrect (G., Quinn-Gregson, Yang and Zenab (2019)).
Nevertheless they had realised the existence and behaviour of minimal ideals in pseudo-finite monoids is important.

A minimal ideal is precisely a **simple** ideal (i.e. a semigroup with no proper ideals).

Simple semigroups are complicated! The 'best' are **completely simple**; such semigroups are isomorphic to Rees Matrix semigroups $\mathcal{M}(G; I, \Lambda; P)$ where G is a group.

Do all pseudo-finite semigroups possess a minimal (completely simple) ideal?

Pseudo-finite semigroups and minimal ideals
General observation

Theorem: G., Miller, Quinn-Gregson and Ruškuc (2022)

A monoid S is pseudo-finite if and only if it has a right ideal that is pseudo-finite as a right S-act.

If I is my right ideal and $u \Rightarrow v$ for all $u, v \in I$, via U, then add $(1, k)$ to U where $k \in I$. Then for any $a, b \in S$ we have

$$a = 1a,\ ka \Rightarrow\ kb, 1b = b.$$

Note that the diameter of S is bounded by diameter of I $+2$.

Consequently, any monoid with a finite minimal ideal is pseudo-finite.

Theorem: GMQ-GR (2022)

A monoid with a minimal completely simple ideal $\mathcal{M} = \mathcal{M}(G; I, J; P)$ is pseudo-finite if and only if it satisfies (A) and (B).

(A) A condition on generating G;

(B) S makes J into a pseudo-finite act.

These conditions are independent.

A semigroup S is **inverse** if S is regular (for all a there exists b with $a = aba$) and idempotents commute.

Theorem: DGQ-GZ (2019)

Let S be an inverse monoid. Then S is pseudo-finite if and only if S has a minimal ideal G where G is a finite group.

Theorem: GMQ-GR (2022)

Let S be a commutative monoid. Then S is pseudo-finite if and only if S has a minimal ideal G where G is a finite group.

Theorem: GMQ-GR (2022)

We extended this to right reversible monoids, where S is **right reversible** if for any $a, b \in S$ we have $c, d \in S$ such that $ca = db$; to completely regular monoids (monoids that are unions of groups) and to orthodox monoids (regular monoids with band of idempotents).

Pseudo-finite semigroups and minimal ideals
Any minimal ideal

The relation $\leq_{\mathcal{J}}$ is defined on S by the rule $a \leq_{\mathcal{J}} b$ if and only if $SaS \subseteq SbS$.

Theorem: GMQ-GR (2022)

Let S be pseudo-finite. Then S has a minimal ideal if and only if S contains an ideal I such that $\leq_{\mathcal{J}} \cap (I \times I)$ is left compatible with multiplication in S.

Corollary: GMQ-GR (2022)

Let S be pseudo-finite such that $\leq_{\mathcal{J}}$ is left compatible. Then S has a minimal ideal.

Theorem: Miller (2020)

The Baer-Levi semigroup BL_X where X is infinite is pseudo-finite, simple, but not completely simple.

Example: GMQ-GR (2022)

Gave an example of an infinite semigroup of mappings that is pseudo-finite but does not have a minimal ideal.

We also developed various techniques to give counter-examples of different kinds.

Diameter
Diagonal acts - again

Diagonal acts

Let S be a semigroup. Then $S \times S$ with action

$$(u, v)t = (ut, vt)$$

is the **(right) diagonal act** of S.

If $S \times S$ is generated by U, then for any $a, b \in S$ we have $(a, b) = (u, v)t$ for some $(u, v) \in U$ and then

$$a = ut \to vt = b, \text{ some } t \in S.$$

Corollary

A semigroup S has diameter 1 if and only if its diagonal act is finitely generated.

Diameter

Corollary: Robertson, Ruškuc & Thompson; Gallagher

For any infinite set X the monoids $\mathcal{T}_X, \mathcal{B}_X$ and $\mathcal{PT}_X$ each has diameter 1.

Let $X = X_1 \dot\cup X_2$ where $|X| = |X_1| = |X_2|$. Let

$$\alpha_i : X \to X_i, \ \ i = 1, 2$$

be bijections. Then if $(\theta_1, \theta_2) \in \mathcal{T}_X \times \mathcal{T}_X$ let

$$\psi = \psi_1 \cup \psi_2 \text{ where } \psi_i : X_i \to X, \psi_i = \alpha_i^{-1}\theta_i$$

giving

$$(\theta_1, \theta_2) = (\alpha_1, \alpha_2)\psi \text{ and so } \mathcal{T}_X = (\alpha_1, \alpha_2)\mathcal{T}_X.$$

Theorem: Gallagher and Ruškuc

For any infinite set X the monoid $\mathcal{I}_X$ does not have diameter 1.

So: as $\mathcal{I}_X$ has a zero,

$\mathcal{I}_X$ has diameter 2.

We are conducting an investigation of diameters of ‘natural’ semigroups of transformations.

Inj_X and BL_X (X an infinite set).

The monoid Inj_X consists of all injective maps $X \to X$.
The Baer-Levi semigroup BL_X consists of all $\alpha \in \mathrm{Inj}_X$ such that $|X \setminus X\alpha| = |X|$.
The semigroup BL_X is right cancellative, right simple and is the minimal ideal of Inj_X.

Theorem: East, G., Miller, Quinn-Gregson, Ruškuc (202?)

The diameter of BL_X is 3.

Since BL_X is the minimal ideal of Inj_X, we have:

Corollary

Let X be infinite. The diameter of Inj_X is no greater than 5.

Theorem: EGMQ-GR (202?)

Let X be infinite. The diameter of Inj_X is 4.

Where to now?

- What are the diameters of other natural transformation semigroups? What about the left diameters?
- Diameters in the context of minimal ideals and other algebraic properties.

Theorem: the team

There is a pseudo-finite regular semigroup with diameter 3 and no minimal ideal.

- Which simple semigroups are minimal ideals of pseudo-finite monoids?
- Homological condition for pseudo-finiteness.
- Can we describe semigroups for which every right congruence of finite index is finitely generated in a bounded way?
- This work is taking place in the general context of finite generation of right congruences...

Thanks!! And references -

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