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Generalised Measurablity and Bilinear forms

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Finite fields

Fact: (Chatzidakis, v.d.Dries, Macintyre)

Let $\phi(x,y) \in \mathcal{L}_{rings}$, then there is a positive constant C and and a finite set $D \subset \{0, 1, ..., |x|\} \times \mathbb{Q}^{>0}$ of pairs (d, μ) such that for each finite field $\mathbb{F}_q$ and each $a \in \mathbb{F}_q^{|y|}$?, if the set $\phi(\mathbb{F}_q, a)$ is non-empty then:

$$||\phi(\mathbb{F}_q, a)| - \mu q^d| \leq C q^{d-\frac{1}{2}}$$

Corollary

The field $\mathbb{F}_p$ cannot be uniformly defined in $\mathbb{F}_{p^2}$

Definition: (Macpherson, Steinhorn, 2008) An $\mathcal{L}$-structure, $\mathcal{M}$ is said to be MS-measurable if there is a function $h = (dim, meas) : Def(\mathcal{M}) \to \mathbb{N} \times \mathbb{R}^{>0} \cup (0,0)$ such that:

Finite	For any $\mathcal{L}$-formula $\phi(x, \bar{y})$ the set $\{h(\phi(x, \bar{a})) : \bar{a} \in M^n\}$ is finite.
Definable	The set of $\bar{a} \in M^n$ such that $h(\phi(x, \bar{a}))$ has a particular value is $\emptyset$-definable.
Singletons	For $\bar{a} \in M^n$, $h(\bar{a}) = (0, 1)$
Additive	Suppose $X, Y \in Def(\mathcal{M})$ disjoint with $dim(X) \geq dim(Y)$ then $dim(X \cup Y) = dim(X)$ and $meas(X \cup Y) = \begin{cases} meas(X) + meas(Y) & \text{if } dim(X) = dim(Y) \\ meas(X) & \text{if } dim(X) > dim(Y) \end{cases}$
Fubini	Let $f : X \to Y$ onto with $h(f^{-1}(y)) = (d, \mu)$ for all $y \in Y$ then $h(X) = (d + dim(Y), \mu meas(Y))$

Examples

- (Chatzidakis, van den Dries, Macintyre) Pseudo finite fields.
- Vector spaces.
- Random graph.

Non-Examples

- ACF.
- $\mathbb{Z}(p^\infty)$.
- SOP: Some $\phi(x, y)$ and $(a_i)_{i \in \omega}$ such that

$$\models \exists x(\phi(x, a_i) \wedge \neg\phi(x, a_j)) \text{ iff } i < j$$

Fact (Macpherson, Steinhorn)

MS-measureable structures are Supersimple finite SU-rank.

Fact (K., Pillay)

Strongly minimal MS-measureable structures are Unimodular

Fact (K., Pillay)

MS-measureable stable structures are One-based

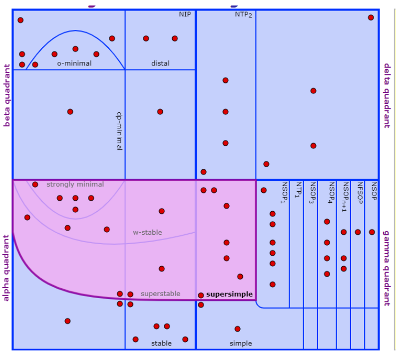


Figure 1: The Universe (see forkinganddividing.com)

Questions

Fields

- Only **known** MS-measurable fields are pseudofinite.
- (Scanlon) MS-measurable fields are quazifinite and perfect, i.e. need PAC!

ω-categorical structures

- (Marimon) Tetrahedron free 3-hypergraph is ω-categorical structures, supersimple rank 1 and one-based, but **not** MS-measurable.
- (Evans, Marimon) Lots of Hrushovski constructions are not MS-measurable.

Measuring semiring (Anscombe, Macpherson, Steinhorn, Wolf)

$T = (T, +, \cdot, 0, 1, <)$ is a measuring semiring if:

- $(T, +, 0)$ and $(T, \cdot, 1)$ are monoids, with $+$ distributing over $\cdot$.
- $\forall x \in T \; x \cdot 0 = 0$
- $(T, <, 0)$ is totally ordered with least element 0.
- $\forall x, y, z \in T$ if $x \leq y$ then $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$.
- For $x, y \in T$ we say the **dimension** of x equals the dimension of y if $x \leq y \leq n \cdot x$ or $y \leq x \leq n \cdot y$ for some $n \in \mathbb{N}$, we write $d(x) = d(y)$.
  $\forall x, y, z \in T$ if $x < y$ and $d(x) = d(z)$ then $x + z < y + z$.

Generalised Measurable (Anscombe, Macpherson, Steinhorn, Wolf)

Given a T measuring semi-ring. An $\mathcal{L}$-structure, $\mathcal{M}$ is said to be T-measurable if there is a function $h =: Def(\mathcal{M}) \to T$ such that:

mac condition	For any $\mathcal{L}$-formula $\phi(x, \bar{y})$ the set $\{h(\phi(x, \bar{a})) : \bar{a} \in M^n\}$ is finite.
Definable	The set of $\bar{a} \in M^n$ such that $h(\phi(x, \bar{a}))$ has a particular value is $\emptyset$-definable.
Finite sets	$h(X) = \lvert X \rvert$ for finite X.
Additive	h is finitely additive.
Fubini	Let $f : X \to Y$ onto with $h(f^{-1}(y)) = t$ for all $y \in Y$ then $h(X) = t \cdot h(Y)$

Examples

- Any MS-measurable structure. T is monomials from $\mathbb{R}[t]$.
- Inf dim vector space over pseudofinite field. $T = \mathbb{R}[t_1, t_2]$.

Non-examples

- algebraically closed fields
- SOP

Pseudofinite bilinear forms (V, F, β)

Take two sorts (V_i, F_i) with $(F_i, +, \cdot, 0, 1)$ a finite field, and $(V_i, +, 0)$ an i–dim vector space over F_i. In two sorted language we also have

- Scalar multiplication: $\lambda : F_i \times V_i \to V_i$.
- Bilinear form: $\beta : V_i \times V_i \to F_i$.

If $|F_i|$ is unbounded we call a non-principal ultraproduct

$$(\mathcal{V}, \mathcal{F}, \beta) = \prod_i (V_i, F_i)/\mathcal{U}$$

an **infinite dimensional vector space over a pseudofinite field with a pseudo-finite bilinear form** and call the common theory T_{bf}^{psf}.

Notation

Fix a monster model $\bar{M} = (\bar{V}, \bar{F})$,

- If X is a set of vectors we use $\langle X \rangle$ to denote the $\bar{F}$-span of X.
- Given A, a subset of $\bar{M}$ we use $A_K = A \cap \bar{F}$ and $A_V = A \cap \bar{V}$
- Given A, a subset of $\bar{M}$, $K_A = (dcl(A))_K$

Facts

- Quantifier elimination when add co-ordinate function and "linear independence" (Granger/Harrison-Shermoen).
- Not simple.
- $NSOP_1$ as Kim-forking is symmetric (Kaplan-Ramsey).
- Generalised measurable in $\mathbb{R}(t_1, t_2)$ (Anscombe-Macpherson-Steinhorn-Wolf).
- Has fine pseudofinite dimension, denoted δ (by above and Garcia-Macpherson-Steinhorn).

Independence relations

Kim-independence

In this structure $A \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} B$ iff

- $acl(A)_K \mathop{\smile\!\!\!\!\!\!\mid}^{F}_{M_K} acl(B)_K$.
- $acl(A)_V \cap acl(B)_V \subseteq M_V$

Pseudo-finite independence

$A \mathop{\smile\!\!\!\!\!\!\mid}^{\delta}_{C} B$ if $\delta(A/C) = \delta(A/BC)$.

These are not the same.

Granger-independence

(Granger) Let $\bar{M} = (\bar{V}, \bar{F}; \beta)$ be a sufficiently saturated model of T. Let $A \subseteq B \subset \bar{M}$ and let $c \in \bar{M}$ (a singleton). We say that $\mathrm{tp}(c/B)$ *does not Γ-fork (dnΓf) over A* if $K_{Ac} \mathop{\smile\!\!\!\!\!\!\;|}^{F}_{K_A} K_B$ and one of the following three conditions holds:

1. $c \in \bar{F}$
2. $c \in \langle A_V \rangle$
3. $c \notin \langle B \rangle$ and $\beta(c, B)$ is Φ-independent over $\beta(c, A)$, i.e. if $b_1, \ldots, b_n \in B_V \setminus \langle A \rangle$ are F-linearly independent then $\{\beta(c, b_1), \ldots, \beta(c, b_n)\}$ is independent, with respect to $\mathop{\smile\!\!\!\!\!\!\;|}^{F}$, over $K_B K_{Ac}$.

If $\mathrm{tp}(c/B)$ does not Γ-fork over A then we write $c \mathop{\smile\!\!\!\!\!\!\;|}^{\Gamma}_{A} B$, and extend this notion to tuples:

$$c_1 \ldots c_n \mathop{\smile\!\!\!\!\!\!\;|}^{\Gamma}_{A} B \text{ iff } c_1 \ldots c_{n-1} \mathop{\smile\!\!\!\!\!\!\;|}^{\Gamma}_{A} B \text{ and } c_n \mathop{\smile\!\!\!\!\!\!\;|}^{\Gamma}_{Ac_1 \ldots c_{n-1}} B$$

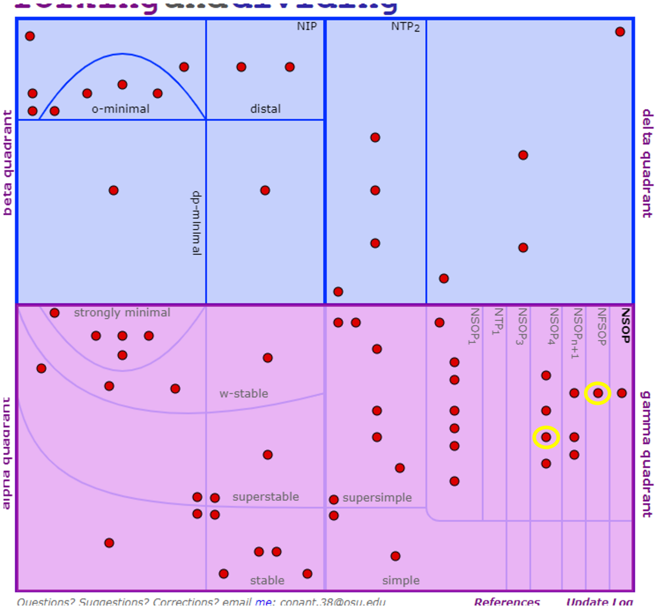
Theorem (needs checking)

$\mathop{\smile}\limits^{\Gamma}$ in T_{bf}^{psf} has:

- Strong finite character.
- Existence over models
- Monotonicity
- Symmetry
- Independent Amalgamation over model
- Extension
- Base monotonicity
- Transitivity

Theorem (needs checking)

$\mathop{\smile}\limits^{\Gamma} = \mathop{\smile}\limits^{\delta}$


NIP
NTP2
o-minimal
distal
dp-minimal
beta quadrant
delta quadrant
strongly minimal
w-stable
superstable
stable
supersimple
simple
NSOP1
NTP1
NSOP3
NSOP4
NSOPn+1
NFSOP
NSOP
alpha quadrant
gamma quadrant
Questions? Suggestions? Corrections? email me: conant.38@osu.edu
References
Update Log

What else?

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Questions

Fields

Do generalised measurable fields coincide with measurable fields?

NSOP hierarchy

How far can we go?