What makes the continuum $\aleph_2$

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Introduction

In this talk I will analyze Cantor's Continuum Problem.

Problem (Cantor)

Determine the cardinality of $\mathbb{R}$*.*

I will first argue in favor of two theses:

Thesis

Two principle goals of Mathematics are to classify and to develop an informative theory.

Thesis

Mathematics is $\mathbb{R}$*-centric.*

We will argue that these favor $|\mathbb{R}| = \aleph_2$.

Mathematics strives to classify and inform

- The classification of surfaces by their genus and end space (Kerékjártó, Richards).
- The Prime Number Theorem (Hadamard).
- The basis theorem for nonplanar graphs (Kuratowski).
- The classification of 3 dimensional space: Geometrization Conjecture (Perelman).
- Mordell Conjecture (Faltings).
- The classification of Bernoulli Shifts by entropy (Ornstein).

Mathematics is $\mathbb{R}$-centric

The fundamental objects of mathematics — the natural numbers, Euclidean space, the complex field, manifolds, operator algebras, algebraic varieties, Lie groups and Lie algebras, partial differential equations — are all fundamentally tied to the real line $\mathbb{R}$.

More general mathematical structures — e.g. groups, rings, and fields — are abstractions of either $\mathbb{R}$ or a closely related structure such as $\mathbb{C}$.

An examination of Hilbert's Problems, the citations of Fields medal winners, the Millennium Problems, etc. yields strong evidence to these two theses.

Even if you are skeptical of these justifications, what is the evidence *against* either of these theses?

The $\mathbf{L}(\mathbb{R})$ absoluteness theorem

The vast majority of mathematical literature actually concerns what is true in the inner model $\mathbf{L}(\mathbb{R})$ — the smallest model of ZF which contains all of the reals and ordinals.
The following theorem is a remarkable remedy to Gödel's incompleteness theorem:

Theorem (Shelah-Woodin)

If there is a supercompact cardinal, then the theory of $\mathbf{L}(\mathbb{R})$ *can not be changed by forcing.*

In fact the theory of $\mathbf{L}(\mathbb{R})$ is largely axiomatized by ZF and the Axiom of Determinacy.
Since independence results in mathematics arising outside of set theory — in algebra, functional analysis, point-set topology — can typically be established using forcing, this suggests that the theory of $\mathbf{L}(\mathbb{R})$ is canonical.

Gödel, on adopting new axioms

...even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success", that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e. consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs.

Gödel, on adopting new axioms

> *...There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even them, as far as that is possible in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.*

This makes a compelling case for accepting the Axiom of Choice.

The Axiom of Choice

Can $\mathbb{R}$ be well ordered?

More generally is the Axiom of Choice true?

Positive answers are fundamentally not $\mathbb{R}$-centric in nature. And yet...

Thesis

The Axiom of Choice has been validated through its utility.

Theorem (Gödel)

If ZF is consistent, then so is ZFC.

Moreover, with the advent of Zorn's Lemma, the Axiom of Choice because useful to the point that most modern mathematicians are uncomfortable working in its absence.

The Axiom of Choice

Some fruitful consequences:

- Every vector space has a basis and any two bases for a vector space have the same cardinality.
- The Hahn-Banach Theorem: every partial bounded linear functional can be extended without increasing its norm.
- A cartesian product of compact spaces is compact.

Even though the full power of the Axiom of Choice it not essential in the most common instances of these theorems, its applicability has led to its widespread acceptance among mathematicians.

The Continuum Hypothesis

The continuum hypothesis is neither $\mathbb{R}$-centric not compatible with classification. The following are consequences of CH:

- (Dusknik-Miller) There is no basis for the uncountable separable linear orders of cardinality less than $2^{2^{\aleph_0}}$.
- (Parovičenko) The measure algebra embeds into $\mathscr{P}(\omega)/\mathrm{fin}$.
- (Philips-Weaver) There is an automorphism of $\mathscr{B}(H)/\mathscr{K}(H)$ which is not inner. Such an automorphism is necessarily not Baire measurable.
- (Todorcevic) There is a c.c.c. topological space whose square is not c.c.c..

Open Graph Axiom

Inspired by work of Abraham-Rubin-Shelah, Todorcevic proposed a combinatorial principle to serve as an alternative to CH:

Open Graph Axiom (OGA)

If G is graph on a separable metric space such that the adjacency relation of G is topologically open, then either G is countably chromatic or G contains an uncountable clique.

Theorem (Todorcevic)

Assume OGA. Every gap in $\omega^\omega/\mathrm{fin}$ is of one of the following types: (ω_1, ω_1^*), (κ, ω^*), or (ω, κ^*) for regular $\kappa \geq \omega_2$.

Open Graph Axiom

Theorem (Farah)

Assume OGA. Every automorphism of $\mathscr{B}(H)/\mathscr{K}(H)$ *is inner.*

Theorem (Dow-Hart)

Assume OGA. The measure algebra does not embed into $\mathscr{P}(\omega)/\mathrm{fin}$.

OGA has the effect of reducing problems in third order arithmetic to problems in second order arithmetic. For instance OGA typically implies that algebraic maps between quotients have liftings with regularity properties.

OGA and the continuum

OGA has a strong influence on the cardinality of the continuum.

Theorem (Todorcevic)

The minimum cardinality of an unbounded subset of $\omega^\omega/\mathrm{fin}$ is $\aleph_2$.

Theorem (M.)

Assume OGA and OCA. The cardinality of $\mathbb{R}$ is $\aleph_2$.

Question (Todorcevic)

Does OGA imply $2^{\aleph_0} = \aleph_2$?

On the other hand, OGA has verifiable consequences:

Theorem (Feng)

OGA is true for open graphs on analytic vertex sets. OGA is a consequence of the Axiom of Determinacy.

Baire Category and Martin's Maximum

In fact OGA is a consequence of a much more general set-theoretic principle, which has roots in classical analysis.

Theorem (Baire)

If K is a compact space, K is not a union of $\aleph_0$ nowhere dense sets.

There are always compact spaces which are a union of $\aleph_1$ nowhere dense sets. For instance, if K is compact and forcing with the open subsets of K, ordered by containment, destroys a stationary subset of ω_1, then K is a union of $\aleph_1$ many nowhere dense sets.

Theorem (Foreman-Magidor-Shelah)

If there is a supercompact cardinal, then there is a forcing extension satisfying Martin's Maximum: if K is an NS_{ω_1}-preserving compact space, then K cannot be covered by $\aleph_1$ nowhere dense sets. In fact $\mathbf{MM}^{++}$ *holds in this extension.*

Consequences of Martin's Maximum

- (Todorcevic) OGA.
- The product of two c.c.c. topological spaces is c.c.c..
- (M.) The uncountable linear orders have a five element basis.
- (Todorcevic) There are five directed systems of cardinality $\aleph_1$ up to Tukey equivalence: 1, ω, ω_1, $\omega \times \omega_1$, $[\omega_1]^{<\omega}$.
- (Foreman-Magidor-Shelah) The nonstationary ideal on ω_1 is ω_2-saturated.
- (Woodin) δ^1_2 is ω_2: there is an effective failure of CH.

Classification theorems for countable linear orders

Theorem

If L is an infinite linear order, then L contains a copy of ω or ω^*.

Theorem (Cantor)

$\mathbb{Q}$ is isomorphic to every $\aleph_0$-dense linear order and universal for the class of all countable linear orders.

Theorem (Laver)

The countable linear orders are well quasi-ordered.

Classification theorems for uncountable linear orders

Theorem (Baumgartner)

Assume MM. Any two $\aleph_1$-dense separable linear orders are isomorphic. In particular if $X \subseteq \mathbb{R}$ has cardinality $\aleph_1$, X embeds into any uncountable separable linear order.

Theorem (M.)

Assume MM. If C is any Countryman line and $X \subseteq \mathbb{R}$ has cardinality $\aleph_1$, then any uncountable linear order contains an isomorphic copy of one of the following linear orders: X, ω_1, $-\omega_1$, C, $-C$.

A linear order is *Countryman* if it is uncountable and yet its square is a countable union of chains. Todorcevic has given explicit recursive constructions of Countryman lines from a ladder system on ω_1.

Classification theorems for uncountable linear orders

Theorem (Abraham-Shelah)

Assume MM. Any two regular Countryman lines are isomorphic or reverse isomorphic.

Theorem (M.)

Assume MM. If C is any Countryman line, then $\eta_C := \mathrm{dirlim}_n(-C + C)^n$ is universal for the class of all Aronszajn lines.

Theorem (Martinez)

Assume MM. The class of Aronszajn lines is well quasi-ordered.

Question (M.)

Is there a classification theorem which entails $2^{\aleph_0} = \aleph_2$? What about the classification of Aronszajn lines?

Verifiable consequences of Martin's Maximum

- The product of two c.c.c. Borel linear orders is c.c.c..
- If K is a c.c.c. Rosenthal compactum, then K is separable.
- Any automorphism of the Calkin algebra with an analytic lifting is inner.
- Borel games are determined.
- OGA for analytic graphs.

Woodin's $\mathbb{P}_{\max}$ extension

Given that $\mathbf{L}(\mathbb{R})$ has a canonical theory but one which violates the Axiom of Choice, it is natural to ask if $\mathbf{L}(\mathbb{R})$ admits an enlargement which satisfies ZFC and has other desirable properties.

Theorem (Woodin)

Assume that there is a supercompact cardinal. There is a homogeneous, σ-closed forcing $\mathbb{P}_{\max}$ in $\mathbf{L}(\mathbb{R})$ such that if $G \subseteq \mathbb{P}_{\max}$ is $\mathbf{L}(\mathbb{R})$-generic:

- $\mathbf{L}(\mathbb{R})[G]$ *satisfies ZFC and* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.
- *If ϕ is a Π_2-sentence in the language $\{\in, \omega_1\}$ and ϕ is true, then* $H_{\aleph_2}^{\mathbf{L}(\mathbb{R})[G]} \models \phi$.
- *If $C \in \mathbf{L}(\mathbb{R})[G]$ is a ladder system on ω_1, then* $\mathbf{L}(\mathbb{R})[G] = \mathbf{L}(\mathbb{R})[C]$.

Woodin's $\mathbb{P}_{\max}$ extension

Since $\mathbb{P}_{\max}$ is homogeneous, the theory of the $\mathbb{P}_{\max}$ extension is invariant under forcing by the $\mathbf{L}(\mathbb{R})$ Absoluteness Theorem.

Thus any Π_2-sentence whose consistency can be established by forcing is forced by $\mathbb{P}_{\max}$ to hold in $H_{\aleph_2}$.

Theorem (Aspero-Schindler)

MM^{++} *implies* $\mathbf{L}(\mathscr{P}(\omega_1))$ *is a* $\mathbb{P}_{\max}$*-generic extension of* $\mathbf{L}(\mathbb{R})$*.*

Concluding remarks

- Martin's Maximum has a multitude of fruitful consequences.
- Martin's Maximum facilitates classification.
- Martin's Maximum provides an $\mathbb{R}$-centric foundation for mathematics.
- MM^{++} provably optimizes classification results at the first cardinality subject to independence phenomena.

Thank you for your attention!