MATTEO VIALE

Absolute Model Companionship, the AMC-spectrum of set theory and the Continuum problem

LEEDS MODELS AND SETS SEMINAR

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① T-ec model ①

FACT TFAE for T, S τ-theories:

(i) $T^{\tau}_{\forall} \supseteq S^{\tau}_{\forall} = \{\psi \in \tau : S \models \psi$ and ψ is universal$\}$

(ii) $\forall \mathcal{M} \left(\mathcal{M} \models T \Rightarrow \exists \mathcal{N} \supseteq \mathcal{M} \ \mathcal{N} \models S \right)$

Def. Let T be a τ-theory

A τ-structure $\mathfrak{M}$ is T-ec :

(i) $\mathfrak{M} \sqsubseteq \mathfrak{N} \models T$

(ii) $\mathfrak{M} \preceq_1 \mathfrak{N}$ if $\mathfrak{N} \models T$ and $\mathfrak{N} \sqsupseteq \mathfrak{M}$

Fact : Let T be a τ-theory TFAE for $\mathfrak{M}$:

(i) $\mathfrak{M}$ is T-ec.

(ii) $\mathfrak{M}$ is $T_\forall^\tau$-ec

$$\begin{tikzcd} \mathfrak{M} \arrow[rr, phantom, bend left, "\preceq_1"] & \sqsubseteq \; \mathfrak{N} \; \sqsubseteq & \mathfrak{P} \\ \models T & \models T_\forall^\tau & \models T \end{tikzcd}$$

Def. Let T be a τ-theory, φ a τ-sentence.

φ is strongly $T^{\tau}_{\forall\vee\exists}$-consistent

$$T^{\tau}_{\forall\vee\exists} = \{\psi : T \models \psi,\ \psi \text{ is bool. comb. of univ. sent.}\}$$

if for all $R \supseteq T$ ~~the~~ $R^{\tau}_{\forall\vee\exists} + \varphi$ is cons.

Fact Assume T is a τ-theory and φ is Π_2-sentence which is strongly $T^{\tau}_{\forall\vee\exists}$-cons. Let $\mathfrak{M}$ be T-ec. Then $\mathfrak{M} \models \varphi$.

Pf Let $\mathfrak{M}_{\#}$ be T-ec ④

Then $R = Th(\mathfrak{M})$ and note that

$$\left(R^{\tau}_{\forall\vee\exists} + \psi\right)_{\forall} \overset{c}{=} \left(R_{\forall\vee\exists}\right)_{\forall}$$

$\psi + R^{\tau}_{\forall\vee\exists}$ is cons.

$\mathfrak{M} \sqsubseteq \mathfrak{N} \models \psi + R^{\tau}_{\forall\vee\exists}$

$\prec_1$

$\Downarrow$

$\mathfrak{M} \models \psi$

$\forall x \exists y\, \varphi(x, y)$

pick $b \in \mathfrak{M}$

then $\mathfrak{N} \models \exists y\, \varphi(x, b)$

$\Downarrow$

$\mathfrak{M} \models \exists y\, \varphi(x, b)$

$\boxtimes$

Question 1

Are the T-ec models axiom. by Π_2-sent. φ which are $T^{\tau}_{\forall\vee\exists}$-cons.?

NO! Counterexample

$T =$ theory of fields in $\{+, \cdot, 0, 1\}$

φ is $\exists x (x^2+1) = 0$

φ is not strongly $T^{\tau}_{\forall\vee\exists}$-cons (it fails in $\mathbb{Q}$)

but ~~if~~ $\mathcal{M}$ is T-ec $\Leftrightarrow$ $\mathcal{M}$ is alg. cl. field.

Model Companionship

Def: Let T, S be τ-theories;
T is model companion of S if:

(i) $T_\forall^\tau = S_\forall^\tau$

(ii) T is model complete

~~M is exact~~ $\mathrm{Mod}(T) = \{M : M \text{ is } S\text{-ec}\}$

Ex: S theory of fields in $\{+, \cdot, 0, 1\}$
T " " ACF

Strong biembeddability

Fact Let T, S be $\mathcal{L}$-theories. TFAE

(i) $\forall \mathcal{M} \models T \; \exists \mathcal{N} \supseteq \mathcal{M} \; \mathcal{N} \models S$

(ii) $S_\forall \subseteq T_\forall$

Question: can we reinforce (i) to

$$\exists \mathcal{N} \supseteq \mathcal{M} \text{ s.t. } Th(\mathcal{N})_{\forall\vee\exists} = Th(\mathcal{M})_{\forall\vee\exists}$$

Note if $\mathcal{M}$ is T-ec and $\mathcal{N} \supseteq \mathcal{M}$ $\mathcal{N} \models T$ then $Th(\mathcal{N})_{\forall\vee\exists} = Th(\mathcal{M})_{\forall\vee\exists}$.

Let $S =$ Fields in $\{+,\cdot,0,1\}$, $\mathbb{Q} \models S$ but $T =$ ACF in $\{+,\cdot,0,1\}$ no $\mathcal{M} \supseteq S$ s.t. $\mathcal{M} \equiv_2 S$

Prop Let T, S be τ-theories TFAE (8)

(i) $T^{\tau}_{\forall\vee\exists} \supseteq S^{\tau}_{\forall\vee\exists}$

(ii) $\forall \mathfrak{M} \models T \; \exists \mathfrak{N} \supseteq \mathfrak{M} \;\; \mathfrak{N} \models S$ and

$$\mathfrak{N} \equiv_1 \mathfrak{M}$$

Def. Let T, S be τ-theories

T is the AMC of S if:

(i) $T^{\tau}_{\forall\vee\exists} = S^{\tau}_{\forall\vee\exists}$

(ii) $\mathrm{Mod}(T) = \{\mathfrak{M} : \mathfrak{M} \text{ is } S\text{-ec}\}$

Fields is the MC of ACF in $\{+, \cdot, 0, 1\}$ but not the AMC.

Thm. Let T, S be τ-theories.

TFAE:

(i) T is the AMC of S

(ii) $Mod(T) = \{\mathcal{M} : \mathcal{M} \text{ is } S\text{-ec}\}$ and $T^{\tau}_{\forall\vee\exists} = S^{\tau}_{\forall\vee\exists}$

(iii) $Mod(T) = \{\mathcal{M} : \mathcal{M} \text{ is } S\text{-ec}\}$ and T is axiomatized by Π_2-sent. φ s.t. φ is $S^{\tau}_{\forall\vee\exists}$-strongly consistent.

Let $S = Th(\mathbb{Z}, \leq)$ $T = Th(\mathbb{Q}, \leq)$ 10

Then T is the AMC of S.

Let S be ~~fields~~ Int. domains and T be ACF

$\{+, \cdot, 0, 1, ^{-1}\}$

$$\forall x \left[(\exists y \; x \cdot y = 1 \Rightarrow x \cdot x^{-1} = 1) \wedge (\neg \exists y (x \cdot y = 1) \Rightarrow x^{-1} = 0) \right]$$

$$(\mathbb{C}, +, \cdot, 0, 1, ^{-1}) \cancel{\preceq_1} (\mathbb{C}[X], +, \cdot, 0, 1, ^{-1})$$

$$\sqsubseteq \quad \cancel{\exists y \neg (y \cdot x = 1)} \quad \exists X (X^{-1} = 0 \wedge X \neq 0)$$

<u>Def</u>: Let τ be a signature
φ a τ-formula

$$Ax^0_\varphi := \forall \vec{x} \left(\varphi(\vec{x}) \Longleftrightarrow R_\varphi(\vec{x}) \right)$$

$$Ax^1_\psi := \forall \vec{x} \left[\left(\exists! y \, \psi(\vec{x}, y) \Rightarrow y = f_\psi(\vec{x}) \right) \wedge \left(\neg \exists! y \, \psi(\vec{x}, y) \Rightarrow f_\psi(\vec{x}) = c_\tau \right) \right]$$

Ax^0_φ is in signature $\tau \cup \{R_\varphi\}$

Ax^1_ψ is in sign. $\tau \cup \{f_\psi, c_\tau\}$

Def. Let $A \subseteq \tau \times \{0,1\}$ be a set of τ-formulae

$$\tau_A = \tau \cup \{R_\varphi : (\varphi, 0) \in A\} \cup \{f_\psi : (\psi, 1) \in A\} \cup \{c_\tau$$

$$T_{\tau,A} = \{Ax_\varphi^t : (\varphi, i) \in A\}$$

Stand Morleyz : $A = \tau \times \{0\}$

Skolem. $A = \tau \times \{1\}$

Def Let T be a τ-theory

The (A)MC-spectrum of T is

$\{A \subseteq \tau \times \{0,1\} : T + F_{\tau,A} \text{ has an (A)MC}\}$

Fact

$-\ \emptyset$ is in $MC(Fields_{\{+,\cdot,0,1\}}) \setminus AMC(Fields_{\{+,\cdot,0,1\}}$

Right signature for set theory (14)

$$\in_{\Delta_0} = \in \cup \{R_\varphi : \varphi \text{ a } \Delta_0\text{-formula}\} \cup$$
$$\{b_\psi : \psi\ \Delta_0 \quad \psi \text{ graph of a Goedel operation}\}$$

<u>Fact</u> $\in_{\Delta_0}$ T_{Δ_0} are T_A for some recursive $A \subseteq \{\in\} \times Z$

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<u>Thm</u> Assume $\tau \supseteq \in_{\Delta_0}$ $T \supseteq ZFC_\tau^- = ZFC + T_{\Delta_0}$ + Repl for τ-formulae

Let S be the MC of T

(i) $S \models ZFC_\tau^{--}$ (a bit of care with Repl.)

(ii) if $\tau \supseteq \in_{\Delta_0} \cup \{K\}$ is s.t.

$\left(H_{K^+}^{\mathcal{M}}, \in_{\Delta_0}\right) \prec_2 \left(\mathcal{M}, \in_\Delta \tau^{\mathcal{M}}\right)$ in all models of T

then $S \models \forall x \exists f (f : K \to x$ is surjection$)$

Thm for ~~there~~ for any definable K 16

there are many $A_K \subseteq E_{\Delta_0} \times \{0,1\}$

s.t $\nexists$ $\cancel{\exists}$ $T_{E_{\Delta_0}, A_K}$ has an AMC.

and A_K is a theory of H_{K^+}

$\neg CH$ is a Π_2-sentence in $\in_{\Delta_0} \cup \{\omega_2\}$

$$\cancel{\forall \alpha (\underbrace{\alpha \subseteq \omega}_{\Delta_0} \Rightarrow)}$$

$$\forall f(\underbrace{f : \omega_2 \to X \text{ is a function}}_{\Delta_0(\omega_2)}) \Rightarrow \exists z(\underbrace{z \subseteq \omega \quad z \notin \mathrm{Im}(f)}_{\Delta_0}$$

Π_2

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<u>Thm</u> Assume $\tau \geq \varepsilon_{10}$ and

$T \supseteq ZFC_\tau$

and $T + \neg CH$ is cons.

Then CH is not in $AMC(T + T_{\tau,A})$

for any A ...

<u>Thm</u> Assume $ZFC + LC$ ~~sat.~~ then

$\exists\, B \in \cancel{AMC(T)}\ Spec_{AMC}(T)$ s.t. $\neg CH \in AMC(T + T_{\tau,B})$