How far is almost strong compactness from strong compactness

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Beyond countable level, many known compactness properties have large cardinal flavors but they are usually follow from strong compactness.

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Definition (Bagaria-Magidor,2014)

Suppose $\kappa \geq \delta$ are uncountable cardinals,

1. κ is δ-*strongly compact* iff every κ-complete filter over X can be extended to a δ-complete ultrafilter over X.
2. κ is *almost strongly compact* iff for any uncountable cardinal $\delta < \kappa$, κ is δ-strongly compact.

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2. κ is *almost strongly compact* iff for any uncountable cardinal $\delta < \kappa$, κ is δ-strongly compact.

Note that κ is κ-strongly compact iff κ is strongly compact, and if a cardinal is greater than the least δ-strongly compact cardinal, then it is also δ-strongly compact.

what does ω_1-strong compactness grant us

The first fact (which is frequently used in showing more sophisticated results) is that ω_1-strong compactness imples SCH above it, in another word Solovay's theorem for strongly compact can be generalized to δ-strong compactness.

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The first fact (which is frequently used in showing more sophisticated results) is that ω_1-strong compactness imples SCH above it, in another word Solovay's theorem for strongly compact can be generalized to δ-strong compactness. Using this, Goldberg proved that the following Conjecture of Woodin holds.

Theorem (Goldberg,2021)

Any two elementary embeddings from the universe into the same inner model agree on the ordinals.

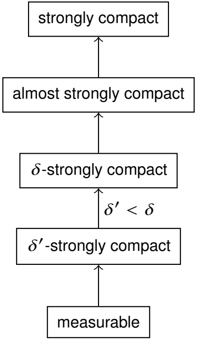
what does ω_1-strong compactness grant us

ω_1-strong compactness implies Woodin's HOD Dichotomy.

Theorem (Goldberg,2021)

Suppose κ is ω_1-strongly compact. Then either all sufficiently large regular cardinals are measurable in HOD or every singular cardinal λ greater than κ is singular in HOD and $\lambda^{\mathrm{HOD}} = \lambda^+$.

Measurable cardinal, δ-strongly compact cardinal, almost strongly compact cardinal and strongly compact cardinal forms a linear hierarchy.



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Proposition

Suppose $\kappa \geq \delta$ are uncountable cardinals. Then the following are equivalent:

1. *κ is δ-strongly compact.*
2. *For any $\lambda > \kappa$, there exists an elementary embedding $j : V \to M$ with M transitive and $\mathrm{crit}(j) \geq \delta$, such that there exists a $D \in M$ with $j''\lambda \subseteq D$ and $M \vDash |D| < j(\kappa)$.*
3. *For every regular λ, there exists a δ-complete uniform ultrafilter over λ.*

Question

If κ is the least ω_1-strongly compact, is it necessarily strongly compact?

Suppose that θ is the first measurable cardinal, then κ is ω_1-strongly compact iff κ is θ-strongly compact.

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By a result of Magidor, the least measurable cardinal may be strongly compact, thus the least ω_1-strongly compact may be strongly compact.

Theorem (Bagaria-Magidor,2014)

Assume κ is a supercompact cardinal, and $\delta < \kappa$ is a measurable cardinal. Then after a suitable Radin forcing, κ is the least δ-strongly compact cardinal and has cofinality δ. Hence κ is not strongly compact cardinal.

Almost strong compactness v.s. strong compactness

In contrast, evidently, almost strong compactness and strong compactness are quite close to each other.

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Theorem (Menas,1974)

If the least almost strongly compact cardinal is measurable,then it is strongly compact.

Theorem (Goldberg,2020)

Assume SCH *holds. Suppose κ is an almost strongly compact cardinal of uncountable cofinality. Then one of the following holds:*

1. *κ is a strongly compact cardinal.*
2. *κ is the successor of a strongly compact cardinal.*
3. *κ is a limit of almost strongly compact cardinals.*

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Corollary (Goldberg,2020)

Assume SCH *holds. If the least almost strongly compact has uncountable cofinality, then it is strongly compact.*

Almost strong compactness v.s. strong compactness

As we can see, there are some subtleties lying between almost strong compactness and strong compactness.

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Question (Boney and Brooke-Taylor)

Is the least almost strongly compact cardinal necessarily strongly compact?

It turns out that Goldberg's theorem no longer holds when the cofinality assumption is dropped.

Theorem (You-Yuan)

Consistently (relative to suitable large cardinal assumptions) the least almost strongly compact cardinal, can be of cofinality ω, and thus it is not necessarily strongly compact

To achieve this (showing that we can have a model which has a least almost strongly compact cardinal with cofinality ω), we need a positive answer to the following question of Bagaria-Magidor:

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Is there a class (possibly proper) $\mathcal{K}$ with $|\mathcal{K}| \geq 2$, and a $\delta_\kappa < \kappa$ for every $\kappa \in \mathcal{K}$, so that κ is the least exactly δ_κ-strongly compact cardinal for every $\kappa \in \mathcal{K}$?

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This is because by a theorem of B-M if κ is the least $\delta < \kappa$ strongly compact then the cofinality of κ has to be strictly larger than δ and thus uncountable.

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This is because by a theorem of B-M if κ is the least $\delta < \kappa$ strongly compact then the cofinality of κ has to be strictly larger than δ and thus uncountable.
If there is a $\mathcal{K}$ of size ω with $\sup_{\kappa \in \mathcal{K}}(\delta_\kappa) = \sup(\mathcal{K})$, then $\sup(\mathcal{K})$ is an almost strongly compact cardinal with SCH holds from below.

Fact

If κ is δ-strongly compact, then for every regular $\lambda \geq \kappa$, every stationary subset $S \subseteq \mathrm{E}^{\lambda}_{<\delta} = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) < \delta\}$ reflects.

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Proof.

Let $j : V \to M$ be an ultrapower map given by a δ-complete ultrafilter over $\mathcal{P}_{\kappa}(\lambda)$.
Let $\beta := \sup(j''\lambda)$ and $T := j(S) \cap \beta$. Then $j''S \subseteq T$.
Meanwhile, for every club C of β in M, let $D := j^{-1}[C]$. Then D is a $< \delta$-club. So there is some $\alpha \in D \cap S \neq \emptyset$. Thus $j(\alpha) \in C \cap j(S) \subseteq C \cap T$. This means that M thinks that T is stationary.
By elementarity, $S \cap \alpha$ is stationary for some $\alpha < \lambda$ of uncountable cofinality. □

Forcing $\mathbb{P}_i$

Gitik extended a basic idea of Kunen's construction of a model with a κ-saturated ideal over an inaccessible cardinal κ.

Theorem (Gitik,2020)

Suppose κ is a supercompact cardinal, $\delta < \kappa$ is a measurable cardinal. Then after a preparation forcing, κ may be the least δ-strongly compact cardinal but not strongly compact after a forcing $Q_{\kappa,\delta}$.

$\mathbb{P}_i$ is the iteration of the preparation forcing and $Q_{\kappa,\delta}$.

$Q_{\kappa,\delta}$

Definition

Let $\kappa > \delta$ be a 2-Mahlo cardinal, define a forcing notion $Q_{\kappa,\delta}$ as follows:

$\langle T, \vec{f} \rangle \in Q_{\kappa,\delta}$ if

1. $T \subseteq {}^{<\kappa}2$ is a normal homogeneous tree of a successor height $\mathrm{ht}(T)$ below κ.
2. $\vec{f} = \langle f_\alpha \mid \alpha < \mathrm{ht}(T) \rangle$ is a δ-ascent path through T, i.e.,
   1. $\vec{f} = \langle f_\alpha \mid \alpha < \kappa, f_\alpha : \delta \to \mathrm{Lev}_\alpha(T) \rangle$.
   2. for every $\alpha, \beta < \kappa$, if $\alpha < \beta$, then the set $\{\gamma < \delta \mid f_\alpha(\gamma) <_T f_\beta(\gamma)\}$ is co-bounded in δ.

The order on $Q_{\kappa,\delta}$ is defined by taking end extensions.

Fact (Gitik,2020)

$Q_{\kappa,\delta}$ adds a pair $\langle T, \vec{f}\rangle$, where T is a δ-ascent κ-Suslin tree, and the function sequence $\vec{f}$ is a δ-ascent path through T.

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Let $i : V \to M$ be the ultrapower map given by a normal measure U over δ. We may lift i to $i^+ : V^{Q_{\kappa,\delta}} \to M^{i(Q_{\kappa,\delta})}$ as $Q_{\kappa,\delta}$ is $< \kappa$-strategically closed. Since T is a κ-Suslin tree, κ is not strongly compact in $V^{Q_{\kappa,\delta}}$.

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But $\vec{f}$ is a δ-ascent path through T, so $\{[f_\alpha]_U \mid \alpha < \kappa\}$ generates a generic cofinal branch of $i(T)$, and the supercompactness of κ in M destroyed by $i(Q_{\kappa,\delta})$ is resurrected by adding the generic cofinal branch.

Lemma

Let $\kappa > \delta$ be regular cardinals. T is a δ-ascent κ-Suslin tree witnessed by $\vec{f}$. Then there is no e.e. $j : V \to M$ with $\mathrm{crit}(j) > \delta$ and $\beta := \sup(j''\kappa) < j(\kappa)$.

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Proof sketch: If not, let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) > \delta$ and $\beta := \sup(j''\kappa) < j(\kappa)$.

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By elementarity and $\mathrm{crit}(j) > \delta$, for any $\alpha < \kappa$,
$\{\gamma < \delta \mid j(\vec{f})_\alpha(\gamma)) <_{j(T)} j(\vec{f})_\beta(\gamma)\}$ is co-bounded.

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Since β has cofinality $\kappa > \delta$, there exists a $\delta' < \delta$ and an unbounded $A \subseteq \kappa$, such that for any $\alpha \in A$, we have for any $\delta' \leq \gamma < \delta$, $j(\vec{f})_\alpha(\gamma)) <_{j(T)} j(\vec{f})_\beta(\gamma)$.

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It's easy to see that $\{\vec{f}_\alpha(\delta') \mid \alpha \in A\}$ generates a cofinal branch through T.

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It's easy to see that $\{\vec{f}_\alpha(\delta') \mid \alpha \in A\}$ generates a cofinal branch through T.

Hence in Gitik's model, κ is exactly δ-strongly compact.

Main idea

Assume $\langle \kappa_i \mid i < \omega \rangle$ is an increasing sequence of supercompact cardinals, and let $\kappa = \lim_{i<\omega} \kappa_i$.
Let $\langle \delta_i \mid i < \omega \rangle$ be an increasing sequence of measurable cardinals, such that $\delta_0 < \kappa_0$ and $\kappa_{i-1} < \delta_i < \kappa_i$ for every $0 < i < \omega$.

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We may find a nice forcing $\mathbb{P}_i$ to make κ_i the least δ_i-strongly compact cardinal.

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We may find a nice forcing $\mathbb{P}_i$ to make κ_i the least δ_i-strongly compact cardinal.
Then take the product forcing $\Pi_{i<\omega}\mathbb{P}_i$, we may make κ_i the least δ_i-strongly compact cardinal for every $i < \omega$. Then κ may be the least almost strongly compact cardinal and has cofinality ω. So κ is not strongly compact.

Theorem (Laver,1978)

Suppose κ is supercompact. Then there is a forcing, say $\mathbb{P}$, so that in $V^{\mathbb{P}}$, κ is still supercompact, and for any $< \kappa$-directed closed forcing $\mathbb{Q}$, κ is also supercompact in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.

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$Q_{\kappa,\delta}$ is $<\delta$-directed closed and $<\kappa$-strategically closed.

We can turn many supercomapct cardinal κ into non-strongly comapct δ-strongly compact cardinal at the same time by using the Eason-product of many $Q_{\kappa,\delta}$.

Theorem (You-Yuan)

Suppose $\mathcal{K}$ is a class of supercompact cardinals containing none of its limit points, and $\mathcal{A} = \langle \delta_\kappa \mid \kappa \in \mathcal{K} \rangle$ is an increasing sequence of measurable cardinals such that for any $\kappa \in \mathcal{K}$, $\delta_\kappa < \kappa$ and $\sup(\mathcal{A} \cap \kappa) < \kappa$. Then in some extension $V^{\mathbb{P}}$, for any $\kappa \in \mathcal{K}$, κ is exactly δ_κ-strongly compact cardinal. Moreover, if $\sup(\mathcal{K} \cap \kappa) < \delta_\kappa$, then κ is the least one. In addition, no new strongly compact cardinals are created.

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This gives an affirmative answer to Bagaria-Magidor's question.

Corollary (You-Yuan)

Suppose the class of supercompact cardinals $\mathcal{K}$ has no measurable limit points, and for any $\kappa \in \mathcal{K}$, $\sup(\mathcal{K} \cap \kappa) < \delta_\kappa < \kappa$ is measurable. Then there is a forcing extension $V^{\mathbb{P}}$, in which for any $\kappa \in \mathcal{K}$, κ is the least δ_κ-strongly compact cardinal. In addition, there is no strongly compact cardinal in $V^{\mathbb{P}}$.

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The case that the order type of $\mathcal{K}$ is ω separates almost strong compactness from strong compactness, which gives a negative answer for the question of Boney and Brook Taylor.

Question

Is it consistent that there exists two singular cardinals $\kappa_0 < \kappa_1$, such that for $i < 2$, κ_i is the least δ_i-strongly compact cardinal for some $\delta_i < \kappa_i$?

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Question

If the least almost strongly compact cardinal is regular, is it necessarily strongly compact?

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