Foundationless geology and a Foundation conservativity result

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The purpose of Foundation

The Axiom of Foundation clarifies the domain of set theory, reducing its study to the well-founded universe V.

It doesn't make sense to ask whether it is "true" since in its absense, it's not clear what set theoretic questions are asking about. E.g., it is a matter of semantics whether sets ought to be allowed to contain themselves as elements.

Conversely, all questions formulated over ZF are about mathematical structure, and may have some claim to a Platonic truth.

Foundation is “mathematically irrelevant”

Since Foundation restricts the domain of set theoretic discourse, we ought to wonder if it inhibits set theory in its role as a foundation of mathematics. Have we lost mathematical structures in this restricton?

With Choice, there is no such concern. From the well-ordering theorem, we know in ZFC – Fund that every mathematical structure is isomorphic to one with an ordinal as its universe.

But is it?

Things are more complicated if you're a Choice skeptic.

Over ZF – Fund, Fund does have mathematical consequences, e.g. "if every orderable set is well-orderable, then every set is well-orderable."

But there may be a choiceless sense in which Fund does not restrict what structures can exist in the universe.

Π_2 sentence classes

To clarify the role of Fund in choiceless mathematics, we will need to examine Π_2 sentences and their interactions with mathematical structure.

Consider the following classes of Π_2 sentences:

- A *rank* Π_2 sentence is of the form "for all α, $V_\alpha \models \varphi$."
- A *structural* Π_2 sentence is of the form "for every structure $\mathcal{M}$ and θ a second-order sentence, $\mathcal{M} \models \theta$."
- The negation of a rank (structural) Π_2 sentence is called a rank (structural) Σ_2 sentence.

Choice example

AC is a structural Π_2 sentence since it is equivalent to every $\mathcal{M}$ satisfying “there exists a total order $<$ on M such that every nonempty subset of M has a minimum.”

Pure AC (that AC holds in the well-founded kernel V) is rank Π_2.

Relationships among Π_2 sentence classes

Rank Π_2 sentences are structural Π_2 sentences: "for all α, $V_\alpha \models \varphi$" can be phrased as "any structure satisfying basic set theory, Σ_2-replacement, and second-order comprehension satisfies $\forall\alpha(V_\alpha \models \varphi)$."

ZF proves these are the same class: a structural Π_2 sentence is true iff every V_α believes it.

ZFC proves all Π_2 sentences are of this form: a Π_2 sentence φ is true iff every V_α satisfies "Σ_2-replacement $\rightarrow \varphi$."

Basic Π_2 conservativity results for Foundation

ZF is conservative over ZF $-$ Fund with respect to rank Π_2 sentences and rank Σ_2 sentences since they live in the well-founded kernel V.

ZF is conservative over ZF $-$ Fund with respect to structural Σ_2 sentences.

Main theorem

ZF is conservative over ZF − Fund with respect to structural Π_2 sentences.

In a sense, this is best possible: "if every orderable set is well-orderable, then every set is well-orderable" is a disjunction of a structural Σ_2 sentence and a structural Π_2 sentence.

In fact, $\mathsf{ZF} + \mathsf{KWP}_1^*$, the assertion that every nonempty set is the surjective image of some $\mathcal{P}(\kappa)$, is also conservative with respect to structural Π_2 sentences. In particular, KWP_1^* is Π_2 but not structural Π_2.

A remark about Jech-Sochor

This theorem can be seen as a generalization of the Jech-Sochor theorem, which says a fixed rank-initial segment of a permutation model of ZFA can be embedded into a symmetric extension of its pure part.

This shows that second-order structures which exist in a permutation model can also exist in a universe of ZF, providing easy consistency proofs for theories like ZF + "there is an amorphous set".

But not all ZFA models are permutation models.

Setting the stage

Working in ZF $-$ Fund, suppose there is a structure $\mathcal{M} \models \theta$, some second-order sentence. We want to find a (proper class) universe of ZF with an isomorphic copy of M.

We can treat the universe of M as a set of atoms. By considering $\mathcal{M}' = (\mathcal{P}^\omega(M), \in)$, we may assume the language of $\mathcal{M}$ is a single binary predicate E and $|M| = |M|^{<\omega}$.

Let $X = (M, \mathcal{P}(M), E)$ and descend to $L(X) \models ZFA +$ "there is a model of θ."

How to proceed

Ultimately, the construction produces a forcing extension of $L(X)$ in which the pure part contains an isomorphic copy of $\mathcal{M}$ with the same power set.

But first we need to understand the geology of $V = V^{L(X)}$. (V^N denotes the pure part of a ZFA model N).

Key geology fact

The remainder of this talk will focus on verifying that V is constructible from a set. By work of Usuba, this implies there is a set of ordinals z such that V is a symmetric extension of $L[z]$.

Small violation of choice

Recall the choice principle SVC(S), which asserts that every nonempty set is for some α the surjective image of $S \times \alpha$. This implies collapsing S to ω forces AC.

For each $x \in L(X)$, let α_x be least such that $x \in L_{\alpha_x+1}(X)$, (φ_x, β_x) least such that there is $p \in X^{<\omega}$ such that
$x = \{s \in L_{\alpha_x} : \varphi_x(s, \beta_x, p)\}$, and
$P_x = \{p \in X^{<\omega} : x = \{s \in L_{\alpha_x} : \varphi_x(s, \beta_x, p)\}$.

Let $S = \{x \in V : \forall y \in V \cap L_{\alpha_x}(X)(P_y \neq P_x)\}$.

Verifying SVC

Fix some κ. We have a surjection $S \times \omega \times \kappa^2$ to $V \cap L_\kappa(X)$ by $(x, n, \alpha, \beta) \mapsto \{s \in L_\alpha(X) : \exists p \in P_x(\varphi_n(s, \beta, p))\}$.

Thus, we have $\mathsf{SVC}(S)$.

Forcing

Let $\mathbb{P} = \mathrm{Coll}(\omega, X \times S)$ and $G \subset \mathbb{P}$ be generic. There is $r \subset \omega$ such that $V^{L(X)[G]} = L[r]$.

In $L(X)[G]$, there is generic $g \subset \mathrm{Coll}(\omega, S)$. Notice $V[g] \models \mathrm{AC}$, and since g is pure, we have $V[g] \subset L[r]$. We will show $V[g]$ is a ground of $L[r]$.

Bukovsky's theorem

Let $\delta = \aleph^*(\mathbb{P})$, i.e. the least ordinal not the surjective image of $\mathbb{P}$.

We will show $V[g]$ uniformly δ-approximates $L[r]$, i.e. that for any κ and $f : \kappa \to \kappa$, $f \in L[r]$, there is $F : \kappa \to \mathcal{P}_\delta(\kappa)$ in $V[g]$ such that for all $\alpha < \kappa$, $f(\alpha) \in F(\alpha)$.

Fix such an $f : \kappa \to \kappa$. Since $f \in L(X)[G]$, it has a name $\dot{f} \in L(X)$. In $L(X)$, define $F(\alpha) = \{\xi < \kappa : \exists p \in \mathbb{P}(p \Vdash \dot{f}(\alpha) = \xi)\}$.

Since F is pure, we have $F \in V \subset V[g]$.

By Bukovsky's theorem, $L[r]$ is a generic extension of $V[g]$, and hence of V as well. Let $\mathbb{Q} \in V$ be such that $L[r] = V^{\mathbb{Q}}$.

Usuba's trick

Consider generic $G \times H \subset \mathbb{Q} \times \mathbb{Q}$. We have $V[G][H] = L[r_1][r_2]$. Each $L[r_i]$ is a ground of $L[r_1][r_2]$, so $L[r_1][r_2]$ has a ground $L[z] \subset L[r_1] \cap L[r_2] \subset V \subset L[r_1]$.

Since $V \models \mathsf{SVC}$, we conclude V is a symmetric extension of $L[z]$.

Some questions

- Working in ZFA, let X be an arbitrary set and N an arbitrary inner model of ZF. Is there $Y \in V^{N(X)}$ such that $V^{N(X)} = N(Y)$?
- Consider $N \models ZFA$ and some generic extension $N[G]$. Is $V^{N[G]}$ a generic extension of V^N?
- On the "non-local" nature of KWP_1^* (and higher such principles): if every $\mathcal{P}^2(\kappa)$ is the surjective image of some $\mathcal{P}(\lambda)$, does that imply KWP_1^*?

Geology references

T. Usuba, “Geology of symmetric grounds.”

R. Schindler, “From set theoretic to inner model theoretic geology.”

Thank you for listening to my talk!