O-minimal tame set-theoretic topology

Pablo Andújar Guerrero

Model and Sets Seminar

Definition

Given a structure $\mathcal{M} = (M, \ldots)$, a topological space (X, τ), $X \subseteq M^n$, is definable in $\mathcal{M}$ if τ has a basis $\mathcal{B}$ that is (uniformly) definable.

Definition

Given a structure $\mathcal{M} = (M, \ldots)$, a topological space (X, τ), $X \subseteq M^n$, is definable in $\mathcal{M}$ if τ has a basis $\mathcal{B}$ that is (uniformly) definable.
I.e. there is a formula $\varphi(\bar{x}, \bar{y})$ such that $\mathcal{B}$ is the family of sets

$$\{\bar{a} : \mathcal{M} \models \varphi(\bar{a}, \bar{b})\} \text{ for } \bar{b} \in M^{|\bar{y}|}.$$

Definition

Given a structure $\mathcal{M} = (M, \ldots)$, a topological space (X, τ), $X \subseteq M^n$, is definable in $\mathcal{M}$ if τ has a basis $\mathcal{B}$ that is (uniformly) definable.

I.e. there is a formula $\varphi(\bar{x}, \bar{y})$ such that $\mathcal{B}$ is the family of sets

$$\{\bar{a} : \mathcal{M} \models \varphi(\bar{a}, \bar{b})\} \text{ for } \bar{b} \in M^{|\bar{y}|}.$$

Examples

Let $\mathcal{M} = (M, <, \ldots)$ expand a dense linear order.

- Euclidean topology (τ_e) on M.

  $\mathcal{B} = \{(b_1, b_2) : b_1 < b_2\}$, $\varphi(x, y_1, y_2) =$ "$y_1 < x < y_2$".

Examples

Let $\mathcal{M} = (M, <, \ldots)$ expand a dense linear order.

- Euclidean topology (τ_e) on M.
- Right half-open interval topology (τ_r) on M (Sorgenfrey line).

  $\mathcal{B} = \{[b_1, b_2) : b_1 < b_2\}$, $\varphi(x, y_1, y_2) =$ "$y_1 \leq x < y_2$".

Examples

Let $\mathcal{M} = (M, <, \ldots)$ expand a dense linear order.

- Euclidean topology (τ_e) on M.
- Right half-open interval topology (τ_r) on M (Sorgenfrey line).
- Left half-open interval topology (τ_l) on M.

  $\mathcal{B} = \{(b_1, b_2] : b_1 < b_2\}$, $\varphi(x, y_1, y_2) =$ "$y_1 < x \leq y_2$".

Examples

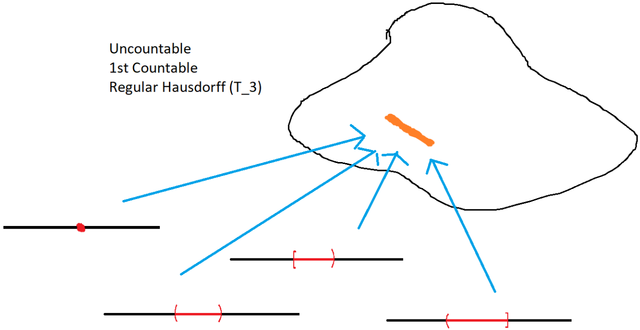
Let $\mathcal{M} = (M, <, \ldots)$ expand a dense linear order.

- Euclidean topology (τ_e) on M.
- Right half-open interval topology (τ_r) on M (Sorgenfrey line).
- Left half-open interval topology (τ_l) on M.
- Discrete topology (τ_s).

  $\mathcal{B} = \{\{b\} : b \in M\}$, $\varphi(x, y) =$ "$x = y$".

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable regular Hausdorff topological space (X, τ), there exists a uncountable set $Y \subseteq \mathbb{R}$ and an embedding $f : (Y, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_s\}$.



Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable regular Hausdorff topological space (X, τ), there exists a uncountable set $Y \subseteq \mathbb{R}$ and an embedding $f : (Y, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_s\}$.

Definition: A linearly ordered structure $\mathcal{M}$ is o-minimal if every definable subset of M is a finite union of points and intervals with endpoints in $M \cup \{\pm\infty\}$, e.g. $\mathcal{M} = (\mathbb{R}, +, \cdot, <)$.

Theorem (AG, Thomas, Walsberg)

Let (X, τ) be an infinite T_1 (singletons are closed) definable topological space in an o-minimal structure $\mathcal{M}$. There exists an interval $I \subseteq M$ and a definable embedding $f : (I, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_l, \tau_s\}$.

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable regular Hausdorff topological space (X, τ), there exists a uncountable set $Y \subseteq \mathbb{R}$ and an embedding $f : (Y, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_s\}$.

Definition: A linearly ordered structure $\mathcal{M}$ is o-minimal if every definable subset of M is a finite union of points and intervals with endpoints in $M \cup \{\pm\infty\}$, e.g. $\mathcal{M} = (\mathbb{R}, +, \cdot, <)$.

Theorem (AG, Thomas, Walsberg)

Let (X, τ) be an infinite T_1 (singletons are closed) definable topological space in an o-minimal structure $\mathcal{M}$. There exists an interval $I \subseteq M$ and a definable embedding $f : (I, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_l, \tau_s\}$.

Proof of Theorem (case $X \subseteq M$)

Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

Proof of Theorem (case $X \subseteq M$)

Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

Key Observation: By o-minimality, for every $A \in \mathcal{B}_x$ there exists $y > x$ such that $(x, y) \subseteq A$ or $(x, y) \cap A = \emptyset$.

Proof of Theorem (case $X \subseteq M$)

Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

Key Observation: By o-minimality, for every $A \in \mathcal{B}_x$ there exists $y > x$ such that $(x, y) \subseteq A$ or $(x, y) \cap A = \emptyset$.

Proof of Theorem (case $X \subseteq M$)

Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

Key Observation: By o-minimality, for every $A \in \mathcal{B}_x$ there exists $y > x$ such that $(x, y) \subseteq A$ or $(x, y) \cap A = \emptyset$.

Proof of Theorem (case $X \subseteq M$)

Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

Key Observation: By o-minimality, for every $A \in \mathcal{B}_x$ there exists $y > x$ such that $(x, y) \subseteq A$ or $(x, y) \cap A = \emptyset$.

Proof of Theorem (case $X \subseteq M$)

- We first show that there exists an interval $J \subseteq X$ such that $\tau_e|_J \subseteq \tau|_J$. That is, for every $x \in X$, $x \in (y, z)$, there is $A \in \mathcal{B}_x$ such that $A \cap J \subseteq (y, z)$. (Non-obvious.)

Proof of Theorem (case $X \subseteq M$)

- We first show that there exists an interval $J \subseteq X$ such that $\tau_e|_J \subseteq \tau|_J$. That is, for every $x \in X$, $x \in (y, z)$, there is $A \in \mathcal{B}_x$ such that $A \cap J \subseteq (y, z)$. (Non-obvious.)
- Then we define two sets $J_1, J_2 \subseteq J$.
    - $J_1 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y > x \, [x, y) \subseteq A\}$.
    - $J_2 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y < x \, (y, x] \subseteq A\}$.

Proof of Theorem (case $X \subseteq M$)

- We first show that there exists an interval $J \subseteq X$ such that $\tau_e|_J \subseteq \tau|_J$. That is, for every $x \in X$, $x \in (y,z)$, there is $A \in \mathcal{B}_x$ such that $A \cap J \subseteq (y,z)$. (Non-obvious.)
- Then we define two sets $J_1, J_2 \subseteq J$.
    - $J_1 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y > x \, [x,y) \subseteq A\}$.
    - $J_2 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y < x \, (y,x] \subseteq A\}$.
- Both J_1 and J_2 are definable. By o-minimality there exists an interval $I \subseteq J$ such that, for $i \in \{1,2\}$, $I \subseteq J_i$ or $I \cap J_i = \emptyset$.

Proof of Theorem (case $X \subseteq M$)

- We first show that there exists an interval $J \subseteq X$ such that $\tau_e|_J \subseteq \tau|_J$. That is, for every $x \in X$, $x \in (y, z)$, there is $A \in \mathcal{B}_x$ such that $A \cap J \subseteq (y, z)$. (Non-obvious.)
- Then we define two sets $J_1, J_2 \subseteq J$.
  - $J_1 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y > x \, [x, y) \subseteq A\}$.
  - $J_2 = \{x \in J : \forall A \in \mathcal{B}_x \, \exists y < x \, (y, x] \subseteq A\}$.
- Both J_1 and J_2 are definable. By o-minimality there exists an interval $I \subseteq J$ such that, for $i \in \{1, 2\}$, $I \subseteq J_i$ or $I \cap J_i = \emptyset$.
- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.

Proof of Theorem (case $X \subseteq M$)

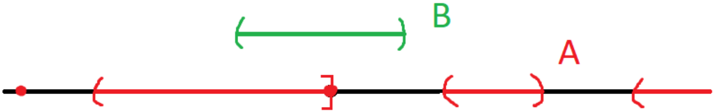
- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.

Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

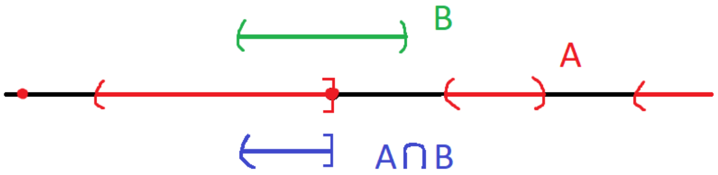
If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.



Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

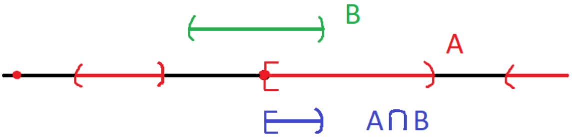
If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.



Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

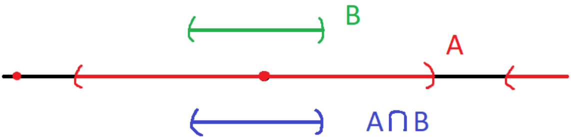
If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.
If $I \subseteq J_1 \setminus J_2 \Rightarrow \tau|_I = \tau_r$.



Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

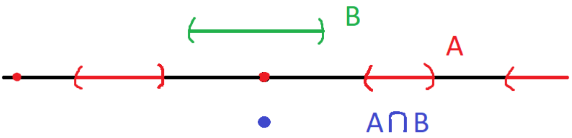
If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.
If $I \subseteq J_1 \setminus J_2 \Rightarrow \tau|_I = \tau_r$.
If $I \subseteq J_1 \cap J_2 \Rightarrow \tau|_I = \tau_e$.



Proof of Theorem (case $X \subseteq M$)

- Finally we show that $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s (restricted to I).

If $I \subseteq J_2 \setminus J_1$ then $\tau|_I$ is the left half-open interval topology τ_l.
If $I \subseteq J_1 \setminus J_2 \Rightarrow \tau|_I = \tau_r$.
If $I \subseteq J_1 \cap J_2 \Rightarrow \tau|_I = \tau_e$.
If $I \cap (J_1 \cup J_2) = \emptyset \Rightarrow \tau|_I = \tau_s$.



Given a class of topological spaces $\mathcal{C}$, a *basis* for $\mathcal{C}$ is a subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that every space in $\mathcal{C}$ contains a homeomorphic copy of a space in $\mathcal{C}_0$.

Given a class of topological spaces $\mathcal{C}$, a *basis* for $\mathcal{C}$ is a subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that every space in $\mathcal{C}$ contains a homeomorphic copy of a space in $\mathcal{C}_0$.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that the class of uncountable first countable regular Hausdorff topological spaces has a 3-element basis given by:

- an uncountable discrete space,
- an uncountable subspace of the reals (with the euclidean topology),
- an uncountable subspace of the Sorgenfrey line.

Given a class of topological spaces $\mathcal{C}$, a *basis* for $\mathcal{C}$ is a subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that every space in $\mathcal{C}$ contains a homeomorphic copy of a space in $\mathcal{C}_0$.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that the class of uncountable first countable regular Hausdorff topological space has a 3-element basis given by:

- an uncountable discrete space,
- an uncountable subspace of the reals (with the euclidean topology),
- an uncountable subspace of the Sorgenfrey line.

Let us denote the condition in blue by (†)

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Stated first in 1986 by Gary Gruenhage in "General topology and its relations to modern analysis and algebra VI: Proceedings of the Sixth Prague Topological Symposium".

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Stated first in 1986 by Gary Gruenhage in "General topology and its relations to modern analysis and algebra VI".

Stated again by Gruenhage in 1990 in "Open problems in Topology", but this time without the first countability condition.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Stated first in 1986 by Gary Gruenhage in "General topology and its relations to modern analysis and algebra VI".

Stated again by Gruenhage in 1990 in "Open problems in Topology", but this time without the first countability condition.

It is known that $(\dagger)$ is false under CH (existence of S-spaces and L-spaces, Roitman 1984)

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Stated first in 1986 by Gary Gruenhage in "General topology and its relations to modern analysis and algebra VI".

Stated again by Gruenhage in 1990 in "Open problems in Topology", but this time without the first countability condition.

It is known that $(\dagger)$ is false under CH (existence of S-spaces and L-spaces, Roitman 1984)

The Proper Forcing Axiom (PFA) implies that there are no S-spaces. Under Martin's Axiom (MA) plus $\neg CH$, there are no first countable L-spaces.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Stated first in 1986 by Gary Gruenhage in "General topology and its relations to modern analysis and algebra VI".

Stated again by Gruenhage in 1990 in "Open problems in Topology", but this time without the first countability condition.

It is known that $(\dagger)$ is false under CH (existence of S-spaces and L-spaces, Roitman 1984)

The Proper Forcing Axiom (PFA) implies that there are no S-spaces. Under Martin's Axiom (MA) plus $\neg CH$, there are no first countable L-spaces.

Moore (2006) proves that the first countability assumption is necessary (construction of an L-space from ZFC).

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Gruenhage (1989) proves that, under PFA, $(\dagger)$ holds for the class of cometrizable spaces (a generalization of metric spaces). Todorcevic (1989) proves it under the Open Coloring Axiom (OCA).

Farhat (2015) observed that, under PFA, $(\dagger)$ holds for monotonically normal (a class that includes both metric and linearly ordered spaces) compacta.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

$ZFC + (\dagger)$ is consistent.

Gruenhage (1989) proves that, under PFA, $(\dagger)$ holds for the class of cometrizable spaces (a generalization of metric spaces). Todorcevic (1989) proves it under the Open Coloring Axiom (OCA).

Farhat (2015) observed that, under PFA, $(\dagger)$ holds for monotonically normal (a class that includes both metric and linearly ordered spaces) compacta.

Refined Conjecture A*

$ZFC + PFA \Rightarrow (\dagger)$.

Published: 25 July 2022

Analysis of a topological basis problem

Y. Peng & S. Todorcevic

Acta Mathematica Hungarica **167**, 419–475 (2022) | Cite this article

Abstract

We examine a basis problem for uncountable regular first countable spaces using the Proper Forcing Axiom. We introduce a notion of inner and outer topologies and show that they come quite close to characterizing the correctness of the current conjecture about this basis problem.

I have two doubts regarding all this.

These are used interchangeably in the literature.

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable T_3 topological space (X, τ), there exists a uncountable subset of the reals with the euclidean, discrete or Sorgenfrey line topology that embeds into it.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that the class of uncountable first countable T_3 topological spaces has a 3-element basis given by an uncountable discrete space, an uncountable subspace of the reals (with the euclidean topology), and an uncountable subspace of the Sorgenfrey line.

These are used interchangeably in the literature.

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable T_3 topological space (X, τ), there exists a uncountable subset of the reals with the euclidean, discrete or Sorgenfrey line topology that embeds into it.

Conjecture A* (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that the class of uncountable first countable T_3 topological spaces has a 3-element basis given by an uncountable discrete space, an uncountable subspace of the reals (with the euclidean topology), and an uncountable subspace of the Sorgenfrey line.

Are these equivalent?

Gruenhage (1989) proves that, under PFA, (†) holds for the class of cometrizable spaces (a generalization of metric spaces).

Gruenhage (1989) proves that, under PFA, (†) holds for the class of cometrizable spaces (a generalization of metric spaces).

Theorem (Gruenhage '89)

Assume PFA. Let (X, τ) be a cometrizable space with no uncountable discrete subspace. Then either

1. (X, τ) contains a copy of an uncountable subspace of the Sorgenfrey line; or
2. X is the continuous image of a separable metric space (i.e. cosmic).

Gruenhage (1989) proves that, under PFA, (†) holds for the class of cometrizable spaces (a generalization of metric spaces).

Theorem (Gruenhage '89)

Assume PFA. Let (X, τ) be a cometrizable space with no uncountable discrete subspace. Then either

1. (X, τ) contains a copy of an uncountable subspace of the Sorgenfrey line; or
2. X is the continuous image of a separable metric space (i.e. cosmic).

Does being cosmic imply containing a copy of an uncountable subspace of the real line?

A related problem

Fremlin's Conjecture

Is it consistent with ZFC that every perfect (no isolated points) Hausdorff compactum admits a continuous and at most two-to-one map onto a metric space?

A related problem

Fremlin's Conjecture

Is it consistent with ZFC that every perfect (no isolated points) Hausdorff compactum admits a continuous and at most two-to-one map onto a metric space?

A positive answer to the 3-element basis conjecture implies a positive answer to Fremlin's conjecture.

Fremlin's conjecture is equivalent, under PFA, to the basis conjecture for subspaces of perfectly normal compacta.

A related problem

Fremlin's Conjecture

Is it consistent with ZFC that every perfect (no isolated points) Hausdorff compactum admits a continuous and at most two-to-one map onto a metric space?

A positive answer to the 3-element basis conjecture implies a positive answer to Fremlin's conjecture.

Fremlin's conjecture is equivalent, under PFA, to the basis conjecture for subspaces of perfectly normal compacta.

Question: Is there an o-minimal definable positive answer to Fremlin's Conjecture?

Going back to o-minimality, we have stronger results for 1-dimensional spaces.

Going back to o-minimality, we have stronger results for 1-dimensional spaces.

Theorem (AG, Thomas, Walsberg)

Let (X, τ) be a Hausdorff definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. There exists a finite partition $\mathcal{I}$ of X into points and intervals such that, for every interval $I \in \mathcal{I}$, the subspace topology $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s.

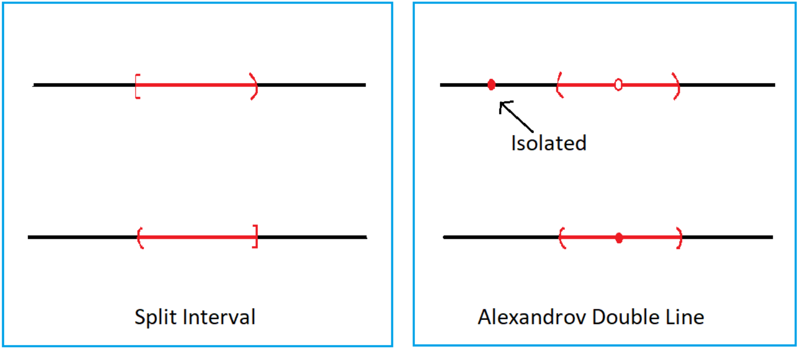
The *split interval* is the set $[0,1] \times \{0,1\}$ with the lexicographic order topology τ_{lex}.

The *Alexandrov double line* is the set $[0,1] \times \{0,1\}$ with the following topology τ_{Alex}:

- All the points in $[0,1] \times \{1\}$ are isolated.
- Basic neighborhoods of points $\langle x, 0 \rangle$ are of the form $(y,z) \times \{0,1\} \setminus \{\langle x, 1 \rangle\}$ for $y < x < z$.

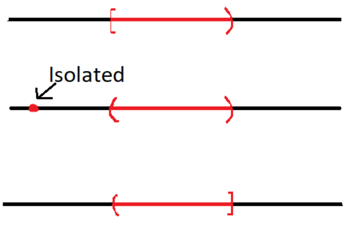
The split interval: $([0,1] \times \{0,1\}, \tau_{lex})$.

The Alexandrov double line: $([0,1] \times \{0,1\}, \tau_{Alex})$.

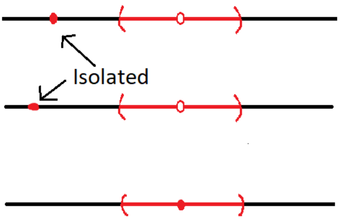


The n-split interval: $(\mathbb{R} \times \{0, \ldots, n-1\}, \tau_{lex})$.

The Alexandrov n-line: $(\mathbb{R} \times \{0, \ldots, n-1\}, \tau_{Alex})$.



n-split interval (n=3)



Alexandrov n-line (n=3)

The n-split interval and the Alexandrov n-line clearly have definable versions in any o-minimal structure $\mathcal{M}$.

The n-split interval and the Alexandrov n-line clearly have definable versions in any o-minimal structure $\mathcal{M}$.

Theorem (AG, Thomas, Walsberg)

Let (X, τ) be a Hausdorff regular definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. There exist disjoint definable open sets Y and Z, with $X \setminus (Y \cup Z)$ finite, and some $n < \omega$, such that

1. There space (Y, τ) embeds definably into the definable n-split interval.
2. There space (Z, τ) embeds definably into the definable Alexandrov n-line.

As a consequence we derive a definable o-minimal positive answer to Fremlin's conjecture, for 1-dimensional spaces.

Corollary

Let (X, τ) be a perfect Hausdorff regular definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. Then (X, τ) admits a definable continuous and at most two-to-one map into (M, τ_e).

Thank you.