Why Mathematical Proof?

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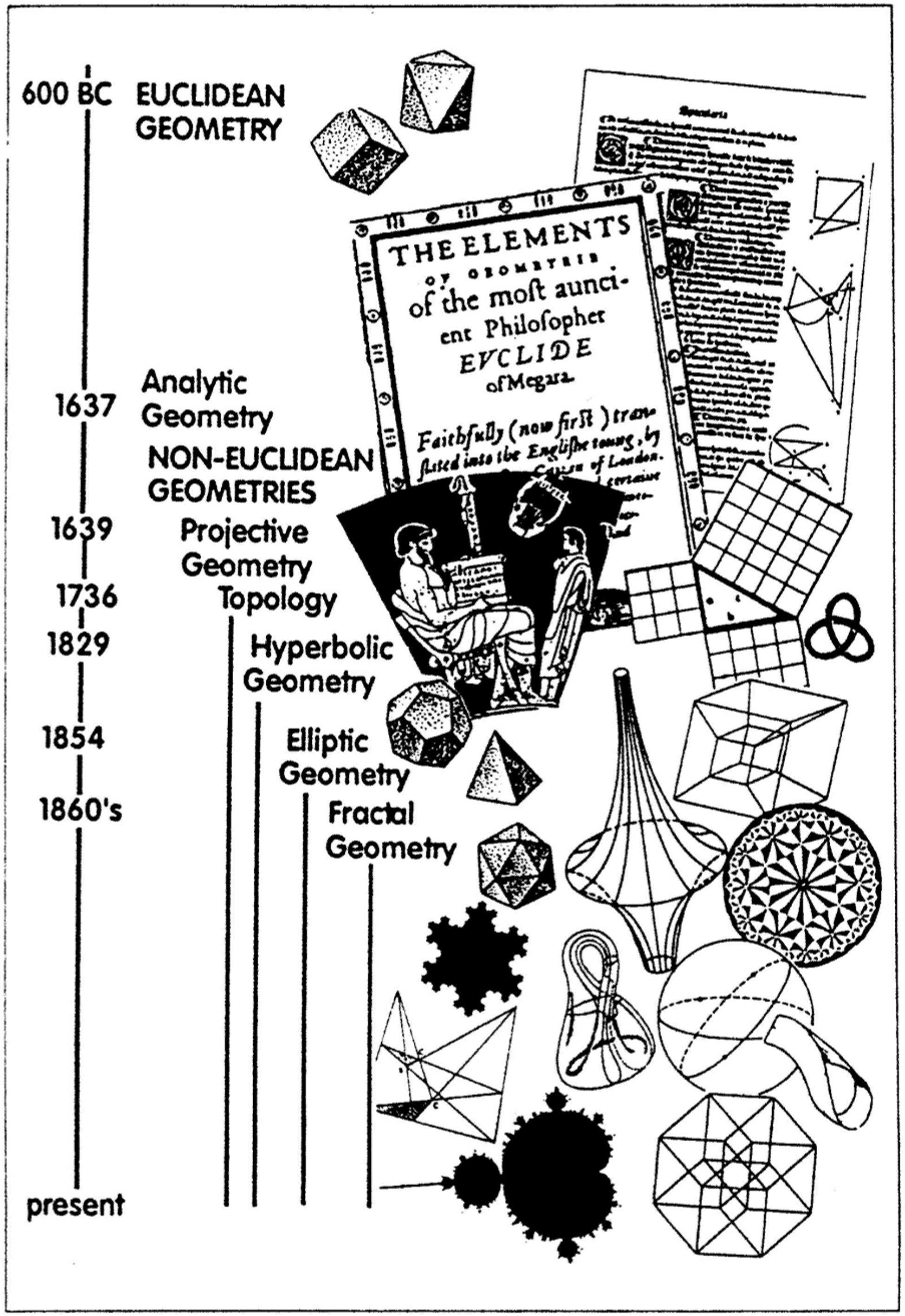
A Timeline for Geometry



Some Greek Geometers

Thales of Miletus (ca. 624 – 548 BC).

Pythagoras of Samos (ca. 580 – 500 BC).

Plato (428 – 347 BC).

Archytas (428 – 347 BC).

Theaetetus (ca. 417 – 369 BC).

Eudoxus of Cnidus (ca. 408 – 347 BC).

Aristotle (384 – 322 BC).

Euclid (ca. 325 – ca. 265 BC).

Archimedes of Syracuse (ca. 287 – ca. 212 BC).

Apollonius of Perga (ca. 262 – ca. 190 BC).

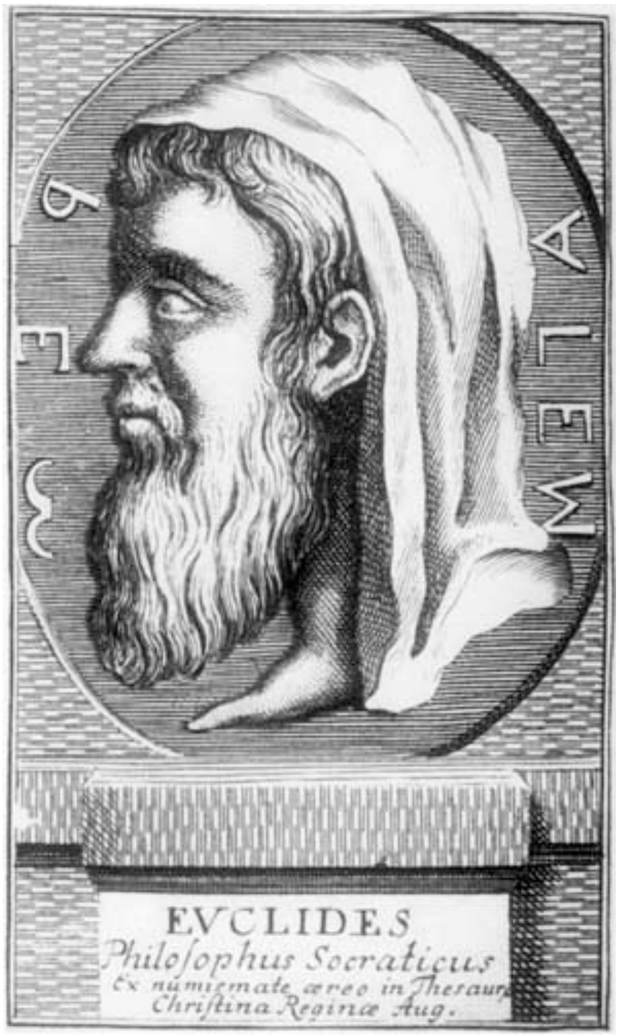
Claudius Ptolemaeus (Ptolemy)(ca. 90 AD – ca. 168 AD).

Diophantus of Alexandria (ca. 200 – 298 AD).

Pappus of Alexandria (ca. 290 – ca. 350 AD).

Proclus Lycaeus (412 – 485 AD).

There is no Royal Road to Geometry



Euclid of Alexandria

ca. 325 — ca. 265 BC

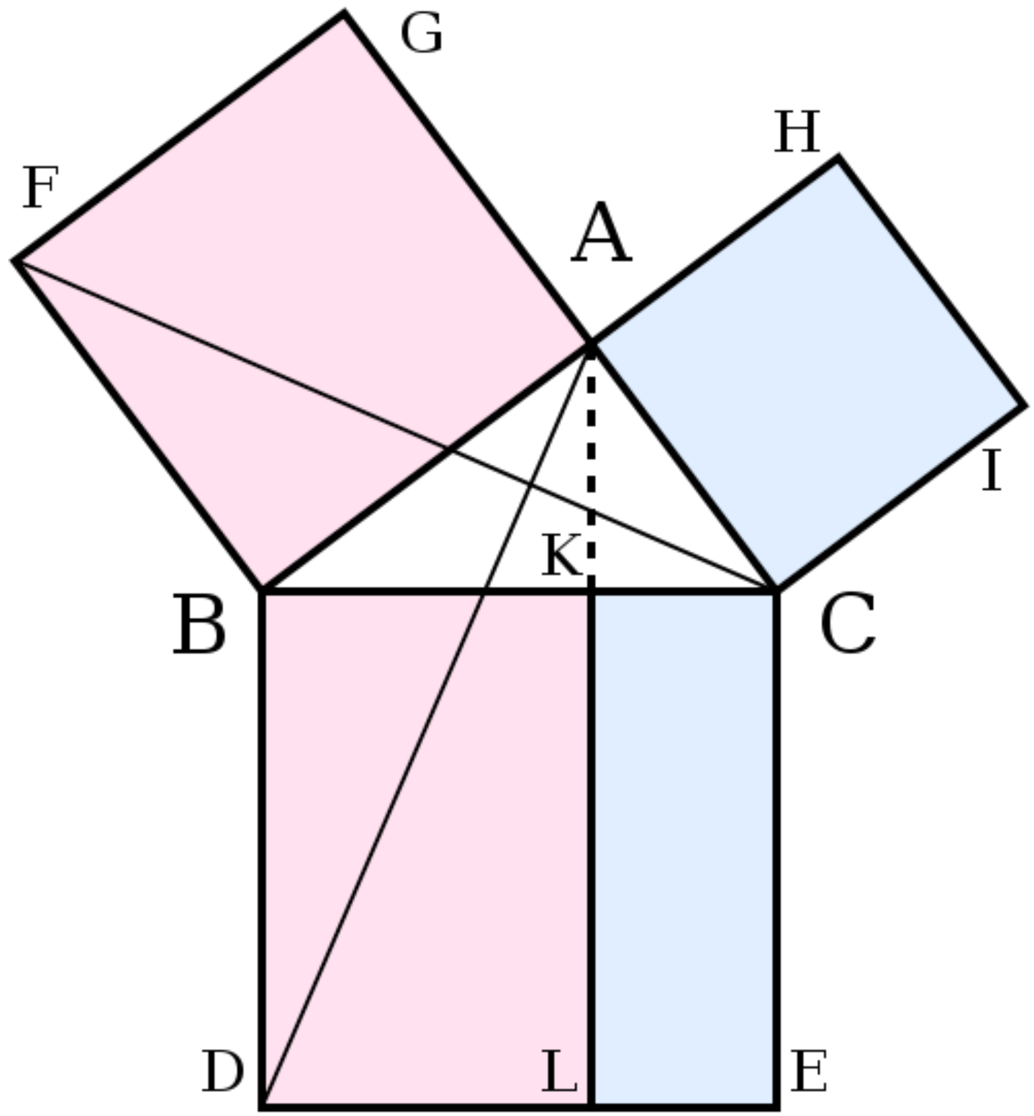
Euclid taught at Alexandria in the time of Ptolemy I Soter, who reigned over Egypt from 323 to 285 BC.

He authored the most successful textbook ever produced — and put his sources into obscurity! Moreover, he made us struggle with proofs ever since.

Why Has Euclidean Geometry Been So Successful?

- Our naive feeling for space **is** Euclidean.
- Its methods have been very **useful**.
- Euclid also shows us a mysterious connection between (visual) **intuition** and **proof**.

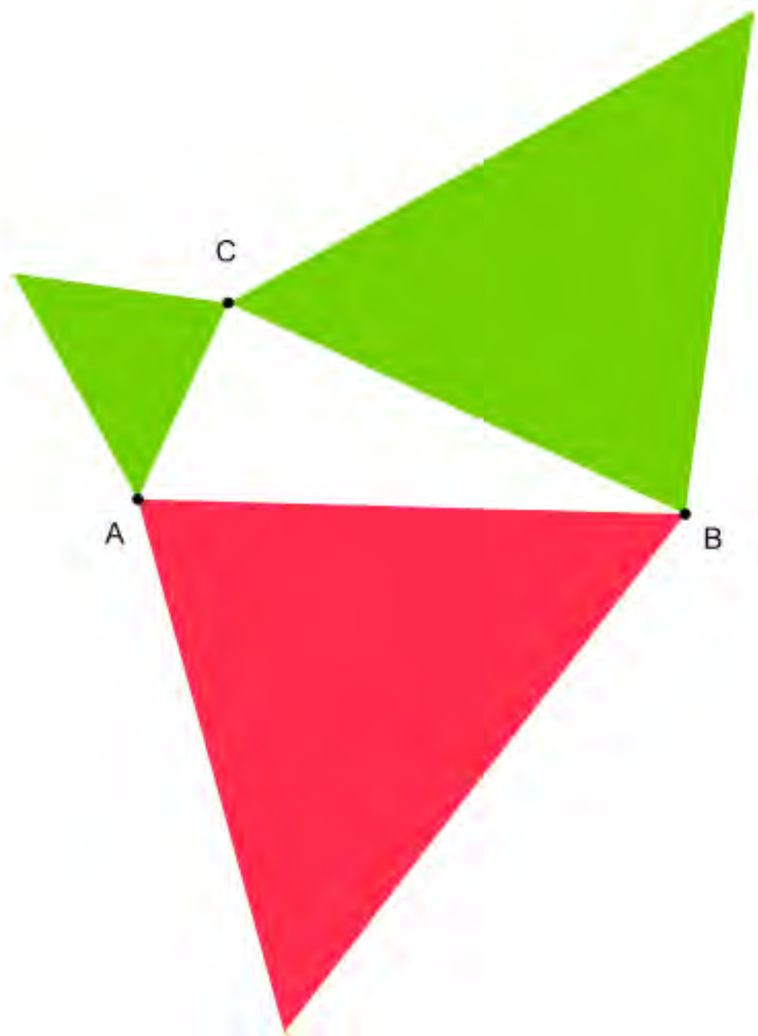
The Pythagorean Theorem



Euclid's Elements: Proposition 47 of Book 1

The Pythagorean Theorem Generalized

Three Similar Figures

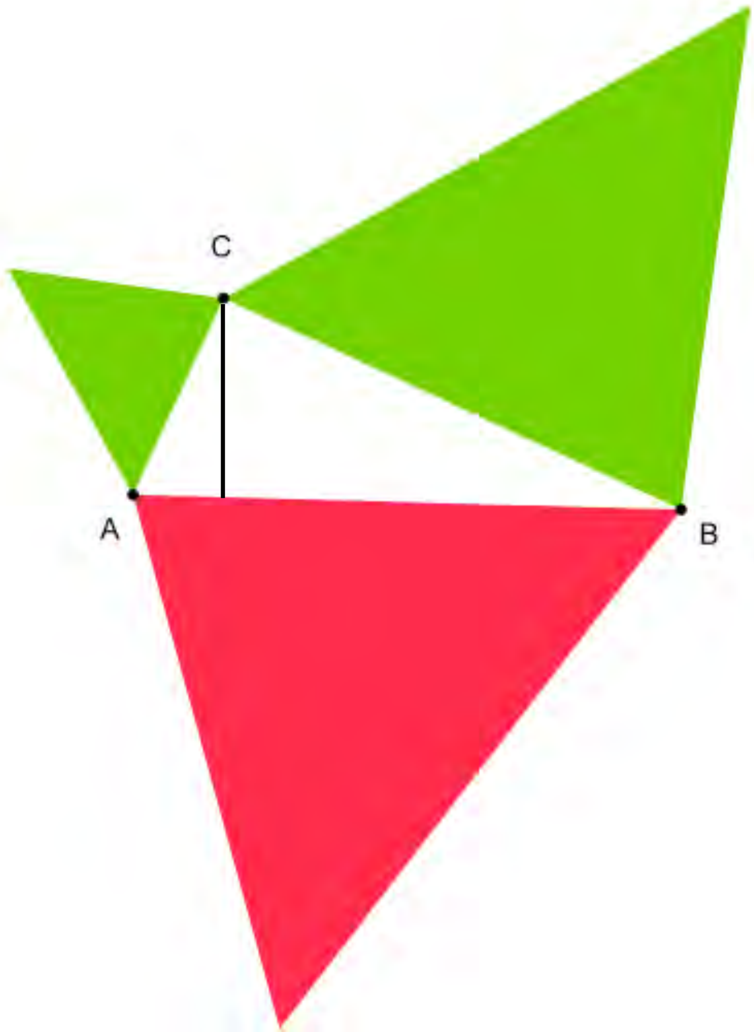


If it holds for one triple, it holds for all.

Now go back and prove the theorem!

The Pythagorean Theorem Generalized

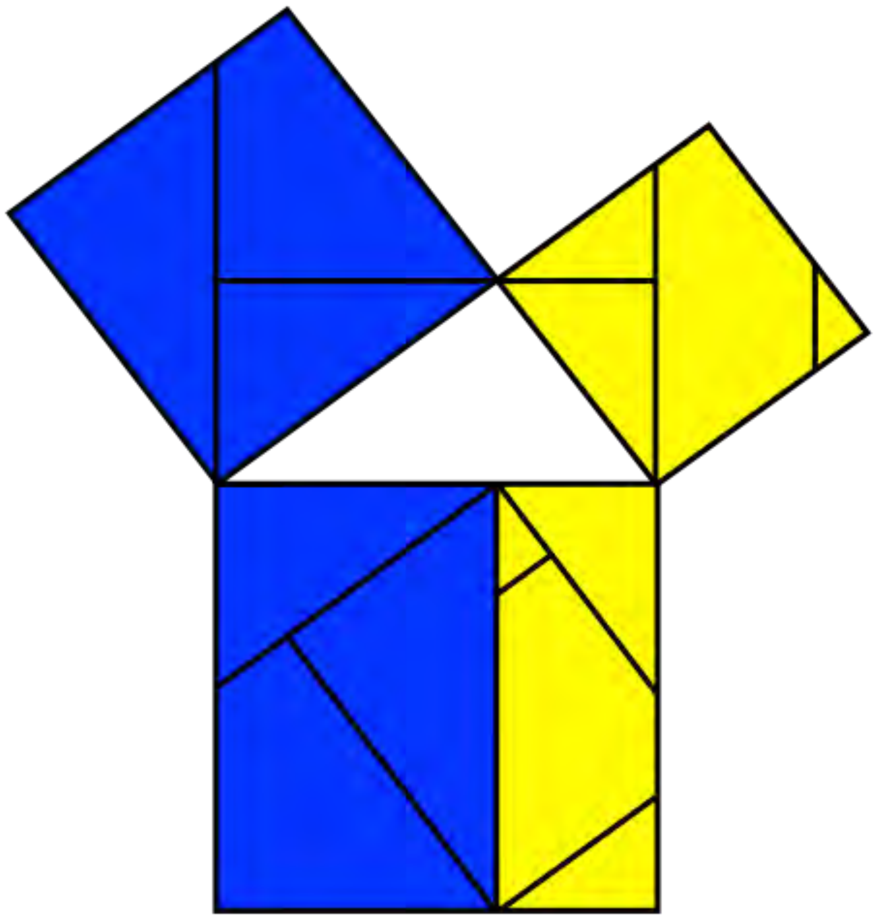
Three Similar Figures



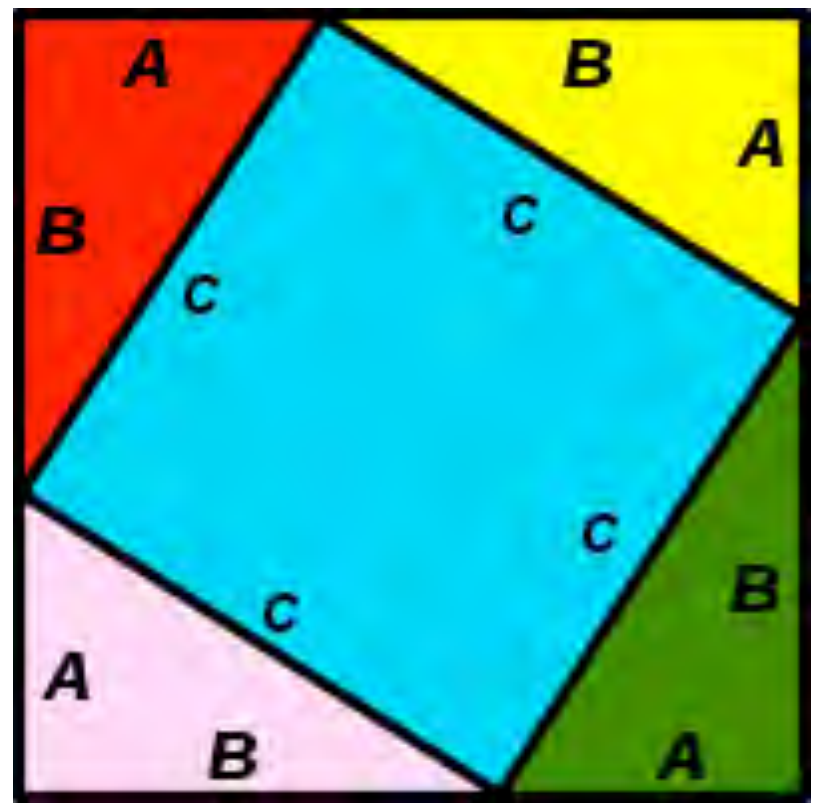
If it holds for one triple, it holds for all.

So, just drop the perpendicular from C.
Now do you see the proof?

Two Other Fine Proofs



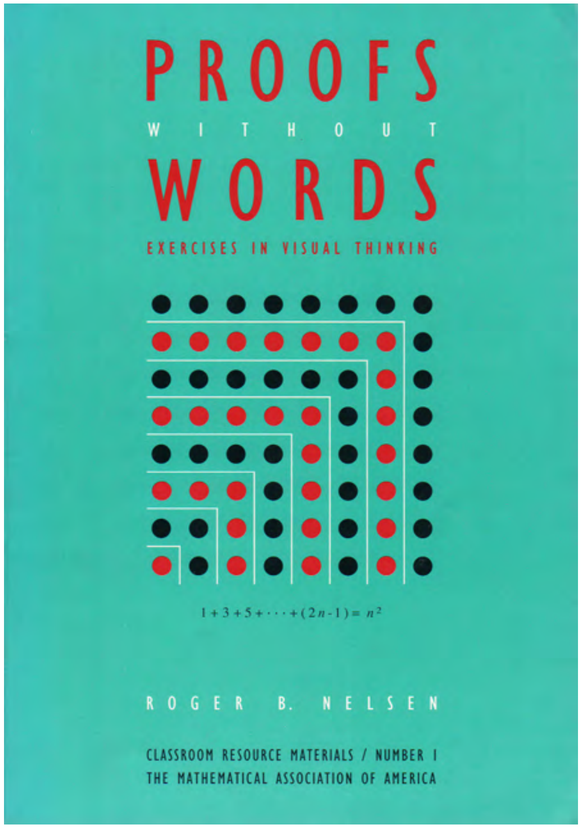
Addition

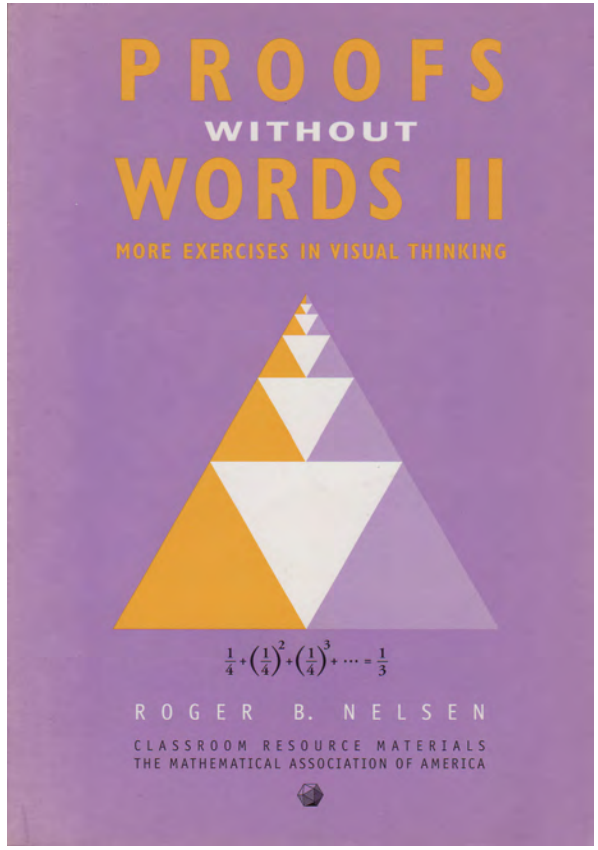


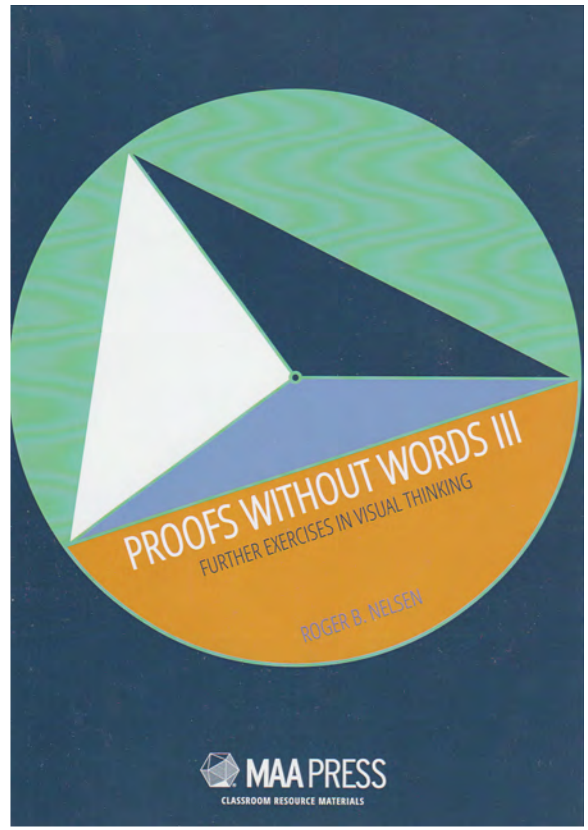
Subtraction

$A^2 + 2\,A\,B + B^2 = 2\,A\,B + C^2$

Three Favorite Books

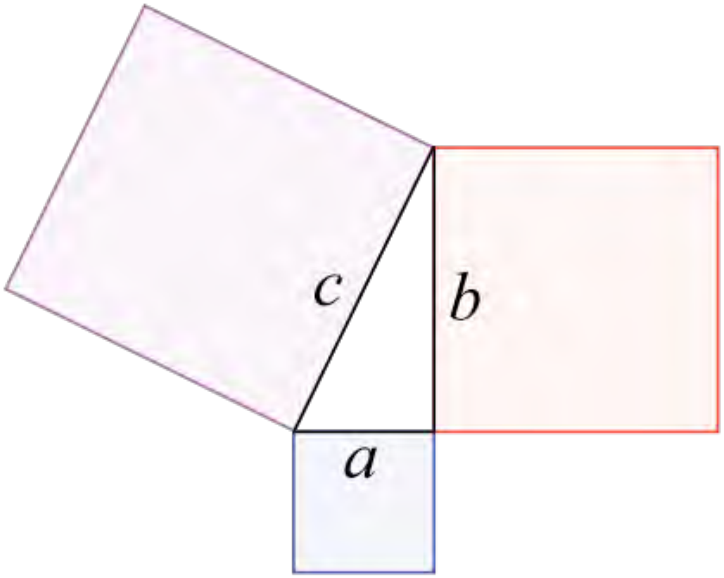






Published by the
Mathematical Association of America

Pythagorean Triples



Question: In a right triangle is it possible to have all edge quantities *integer*?

$$a^2 + b^2 = c^2$$

There are 16 primitive Pythagorean triples with c ≤ 100:

(3, 4, 5) (5, 12, 13) (8, 15, 17) (7, 24, 25)

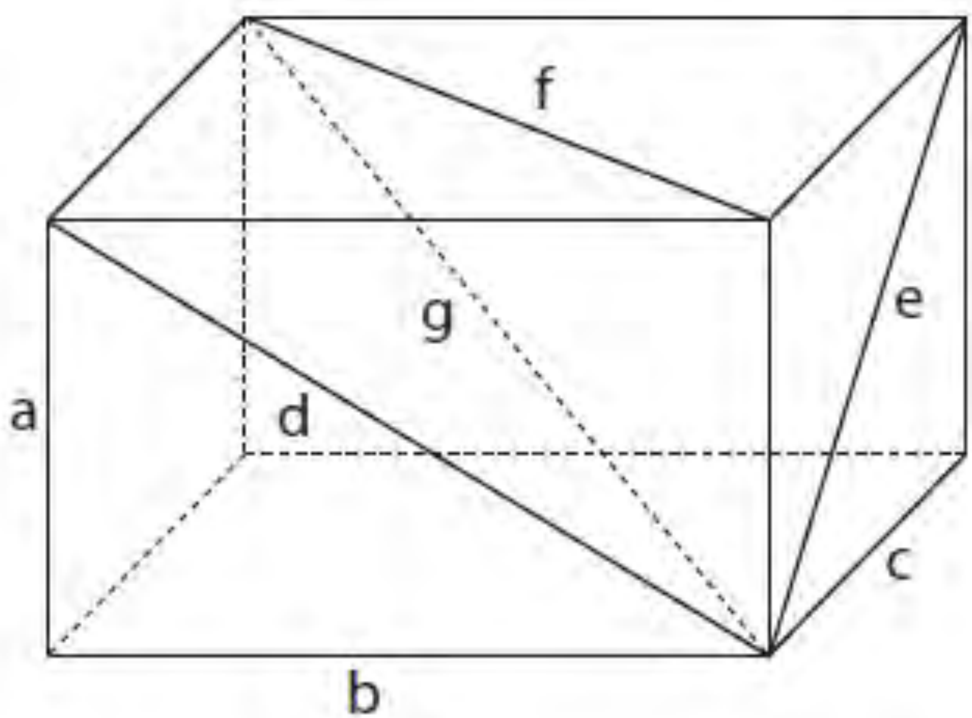
(20, 21, 29) (12, 35, 37) (9, 40, 41) (28, 45, 53)

(11, 60, 61) (16, 63, 65) (33, 56, 65) (48, 55, 73)

(13, 84, 85) (36, 77, 85) (39, 80, 89) (65, 72, 97)

Euler's Brick

$$
\begin{aligned}
a^2 + b^2 &= d^2 \\
a^2 + c^2 &= e^2 \\
b^2 + c^2 &= f^2 \\
a^2 + b^2 + c^2 &= g^2.
\end{aligned}
$$



Question: Is it possible to have all those quantities *integer?*

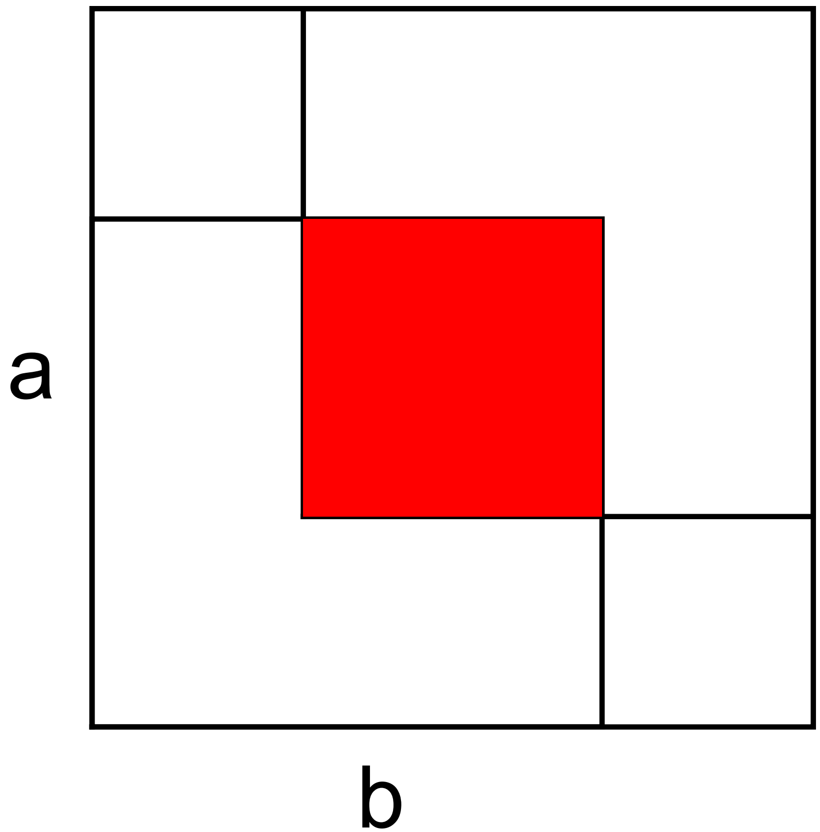
Answer: It is still *unknown!* But it is possible to have *all but* g integer.

Smallest solution (1719):
a = 117, b = 240, c = 44.

Exhaustive computer searches show that, if a perfect brick exists, one of its edges must be greater than 3×10^{12}, and its smallest edge must be longer than 10^{10}.

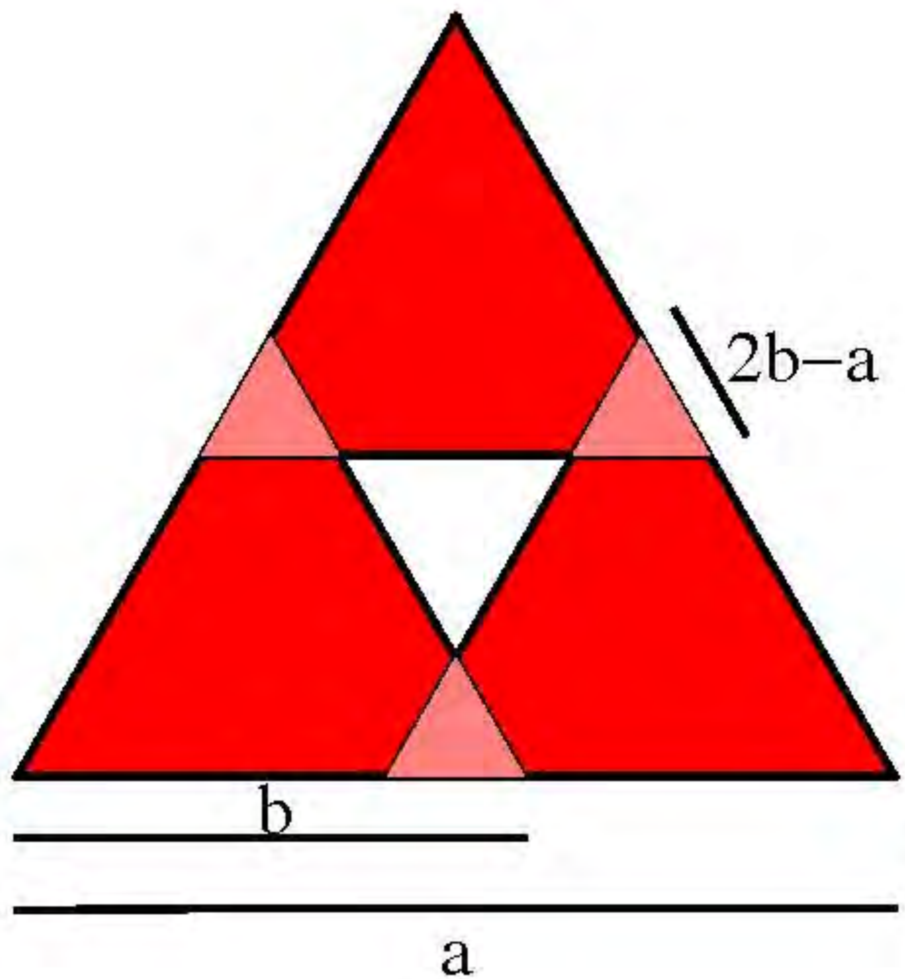
Is √2 Rational or Irrational?

Does $2b^2 = a^2$ have an integer solution?



Stanley Tennenbaum's suggestion: Draw a diagram of squares using the smallest possible integers. Hmmmm. . .

Proof of the Irrationality of √3



Irrationality from the Book
by
Steven J. Miller and David Montague

Do We Really Need Proofs?

Logic, Padoa says, is not in
a particularly fortunate position.
On the one hand,
philosophers prefer to speak of it
without using it ;
while on the other hand
mathematicians prefer to use it
without speaking of it —
and even without desiring *to hear it spoken of*.

— *C.H. Langford (1937)*

Peter Lee's Favorite Fortune Cookie

That which must be proved
cannot be worth much!

My Favorite Fortune Cookies

From listening comes wisdom;
from speaking, repentance.

A problem clearly stated is a problem half solved.

How Should We Do Mathematics?

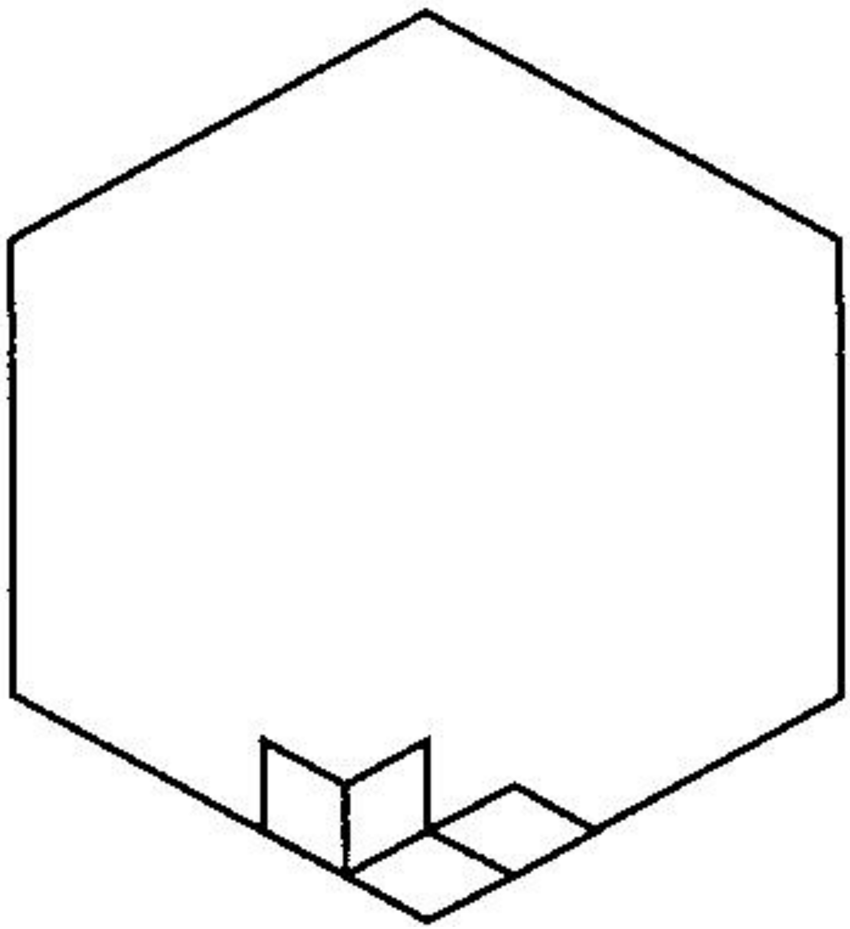
Mathematics is about ideas which ***explain*** and thus enable us to ***understand*** things.

The solution of a mathematical problem is ***beautiful*** in proportion to the beauty of the idea used to solve it.

So, appreciating mathematics means learning to recognize and appreciate ***beautiful ideas***....

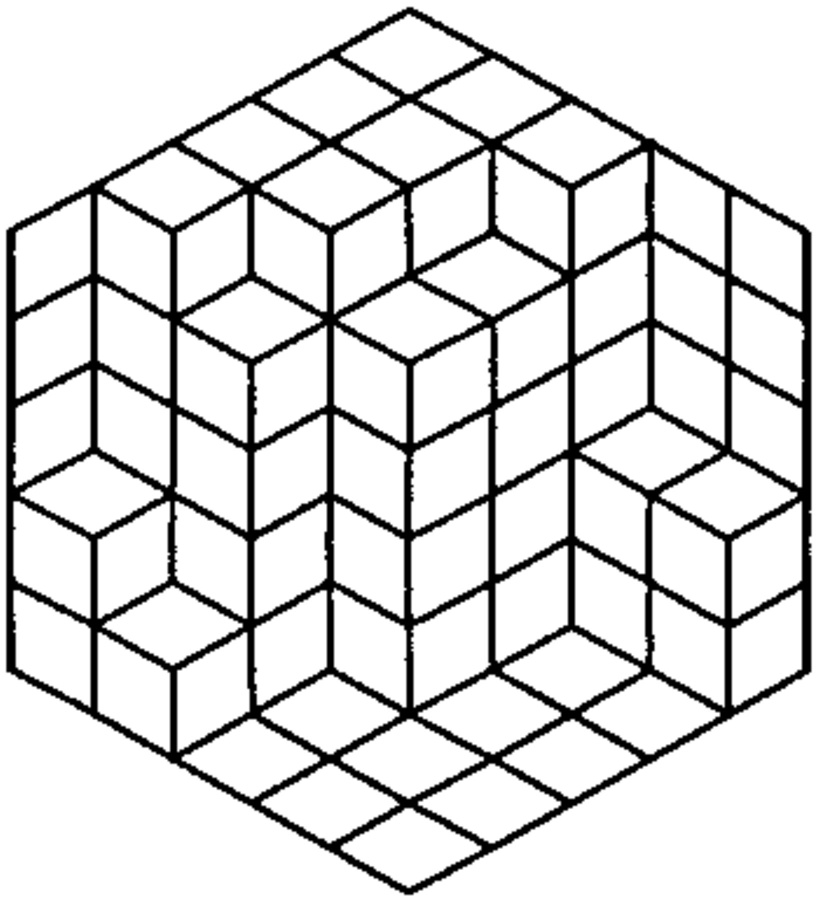
— David Gale (1921 – 2008)

How can the box be filled with rhombi?



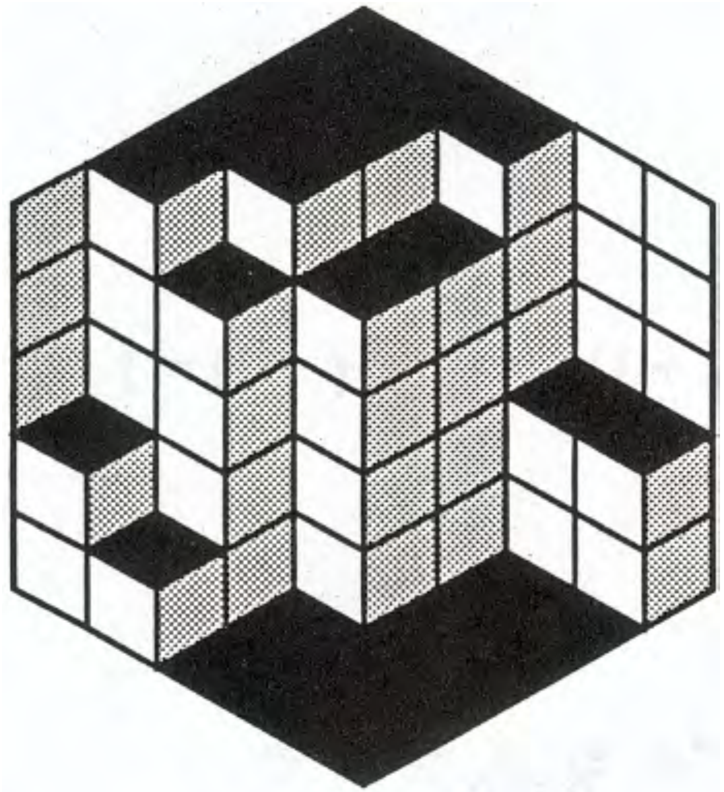
Note the three orientations.

However you do it, there are the same number of each orientation!



Why?

Give Each Orientation a Color



Is this a proof?

G. David and C. Tomei. “The problem of the calissons, “ American Mathematical Monthly, vol. 96 (1989), pp. 429–431.

The late **Prof. Dr. Edsger W. Dijkstra** in his handwritten, privately circulated note, EWD 1055, of 5 July, 1989, strongly rejected this method of argument!

The late **Prof. N.G. de Bruijn** has also written on this problem at: `alexandria.tue.nl/repository/freearticles/599882.pdf`.

Comprehending the Universe

Philosophy is written in that great book which ever lies before our eyes — I mean the Universe — but we cannot understand it if we do not first learn the ***language*** and grasp the ***symbols*** in which it is written.

This book is written in the ***mathematical language***, and the symbols are triangles, circles, and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

— Galileo Galilei (1564 – 1642)

Atiyah Plucks the Strings

"The mathematics involved in string theory is quite remarkable by any standards. In subtlety and sophistication it vastly exceeds previous uses of mathematics in physical theories.

Almost every part of contemporary mathematics is involved somewhere in the story. Even more remarkable is that string theory has led to a ***whole host of amazing results in mathematics*** in areas that seem far removed from physics.

To many this indicates that string theory must be ***on the right track***. ... Time will tell."

— *Sir Michael Atiyah* (*Nature* for December, 2005)

If you haven't found something strange during the day, it hasn't been much of a day.

— *John Archibald Wheeler (1911-2008)*

Those Mathematicians!

Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.

Mathematicians are a kind of Frenchmen: you tell them something, they translate into their own language, and then it immediately means something else.

J. W. von Goethe
(1749 – 1832)

An Old Joke

A physicist and an engineer are in a hot-air balloon.
Soon, they find themselves lost in a canyon somewhere.
They yell out for help:

"Helllloooooo! Where are we?"

Fifteen minutes later, they hear an echoing voice:

"Helllloooooo! You're in a hot-air balloon!"

The physicist says, ***"That must have been a mathematician."***

The engineer asks, ***"Why do you say that?"***

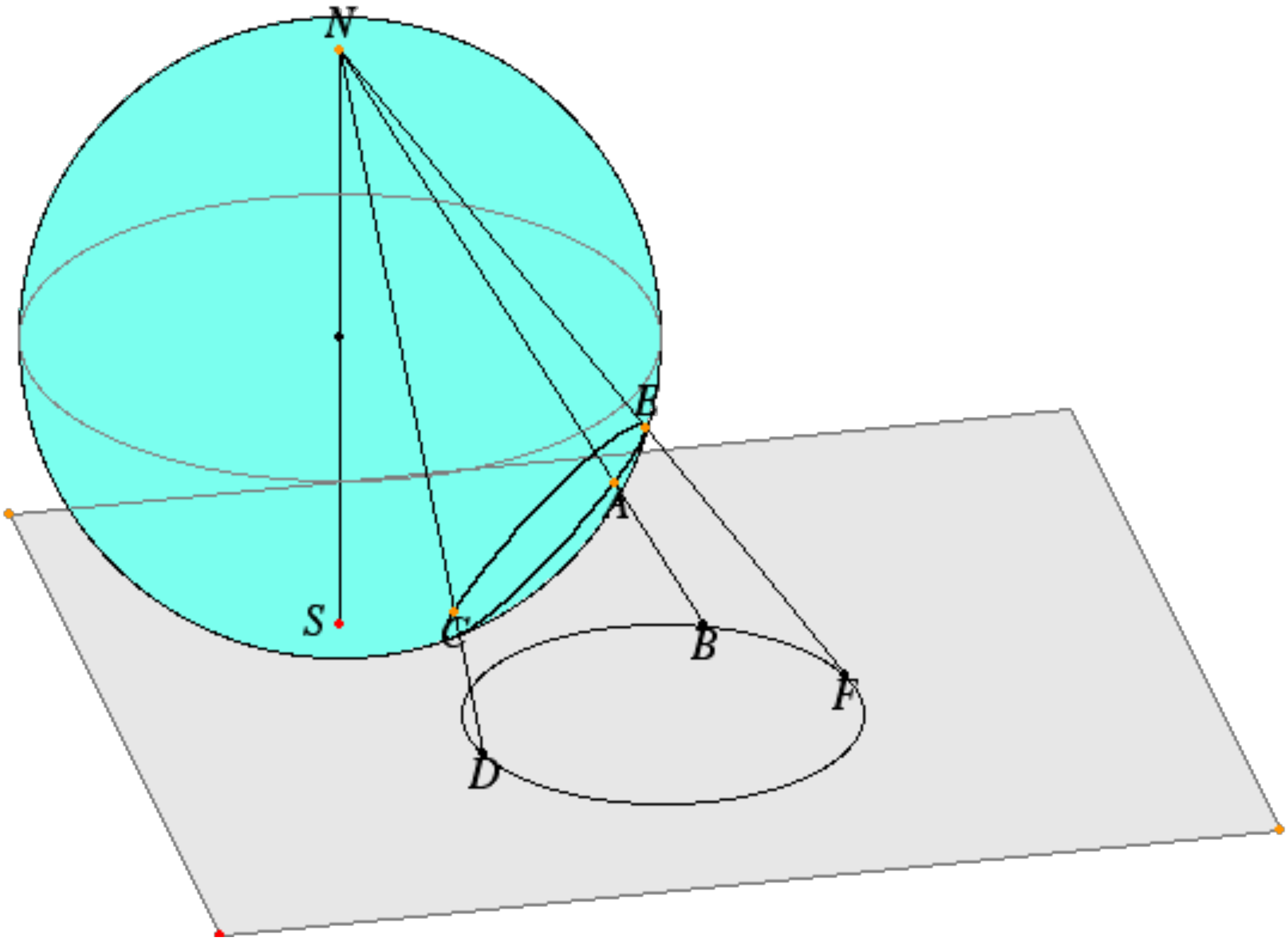
The physicist replies, ***"The answer was absolutely correct and utterly useless."***

Stereographic Projection of the Sphere. I

Stereographic projection was known to Hipparchus, Ptolemy and probably earlier to the Egyptians. It was originally known as the planisphere projection. *Planisphaerium* by Ptolemy is the oldest surviving document that describes it. One of its most important uses was the representation of celestial charts.

The term *planisphere* is still used to refer to such charts.

Stereographic Projection of the Sphere. II

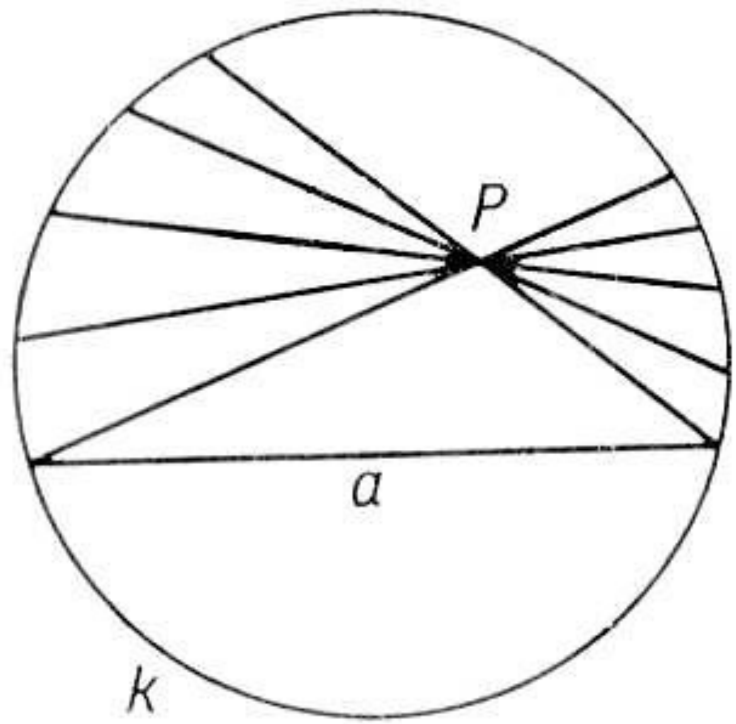


Note: Circles project to circles and angles are preserved!

Question: How to find the **center** of the lower circle?

Stereographic Projection of the Sphere. III

Two Models of Hyperbolic Geometry

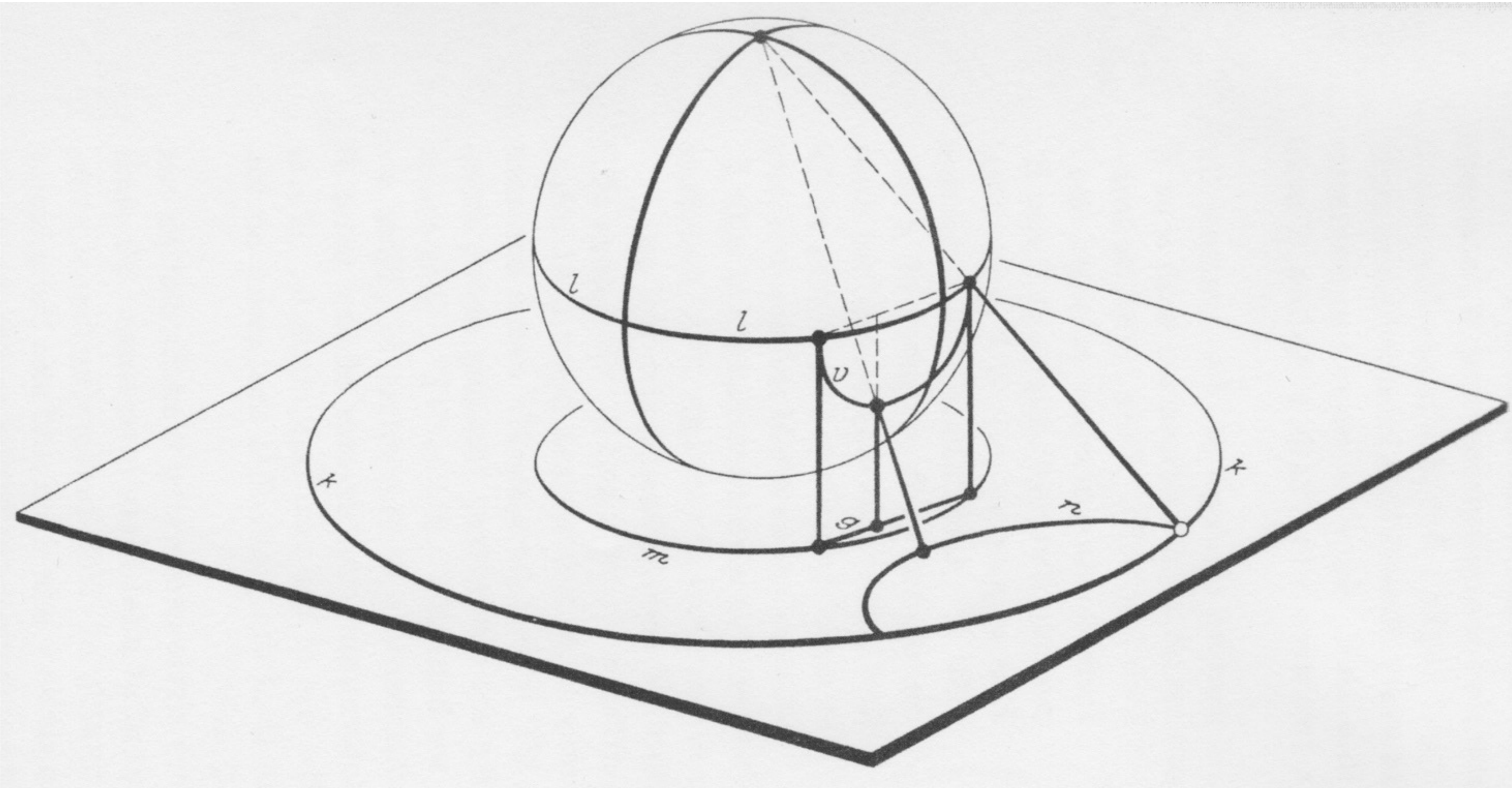


Eugenio Beltrami
(1835 – 1900)
Felix Christian Klein
(1849 – 1925)

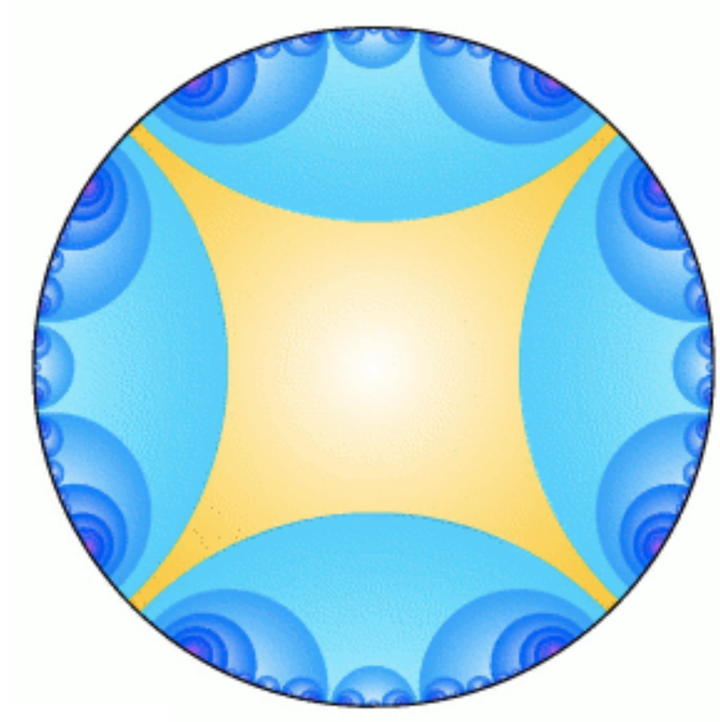
Jules Henri Poincaré
(1854 – 1912)

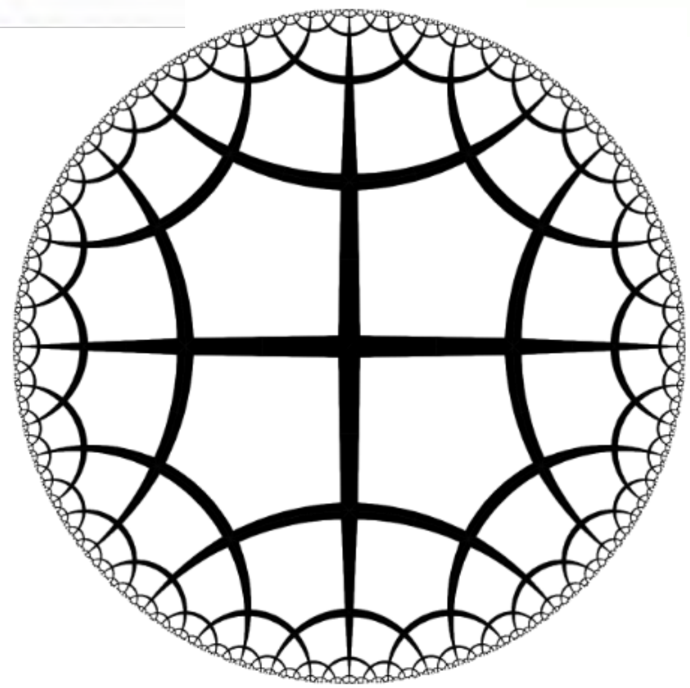
This model has correct angles!

The Models Are Isomorphic!

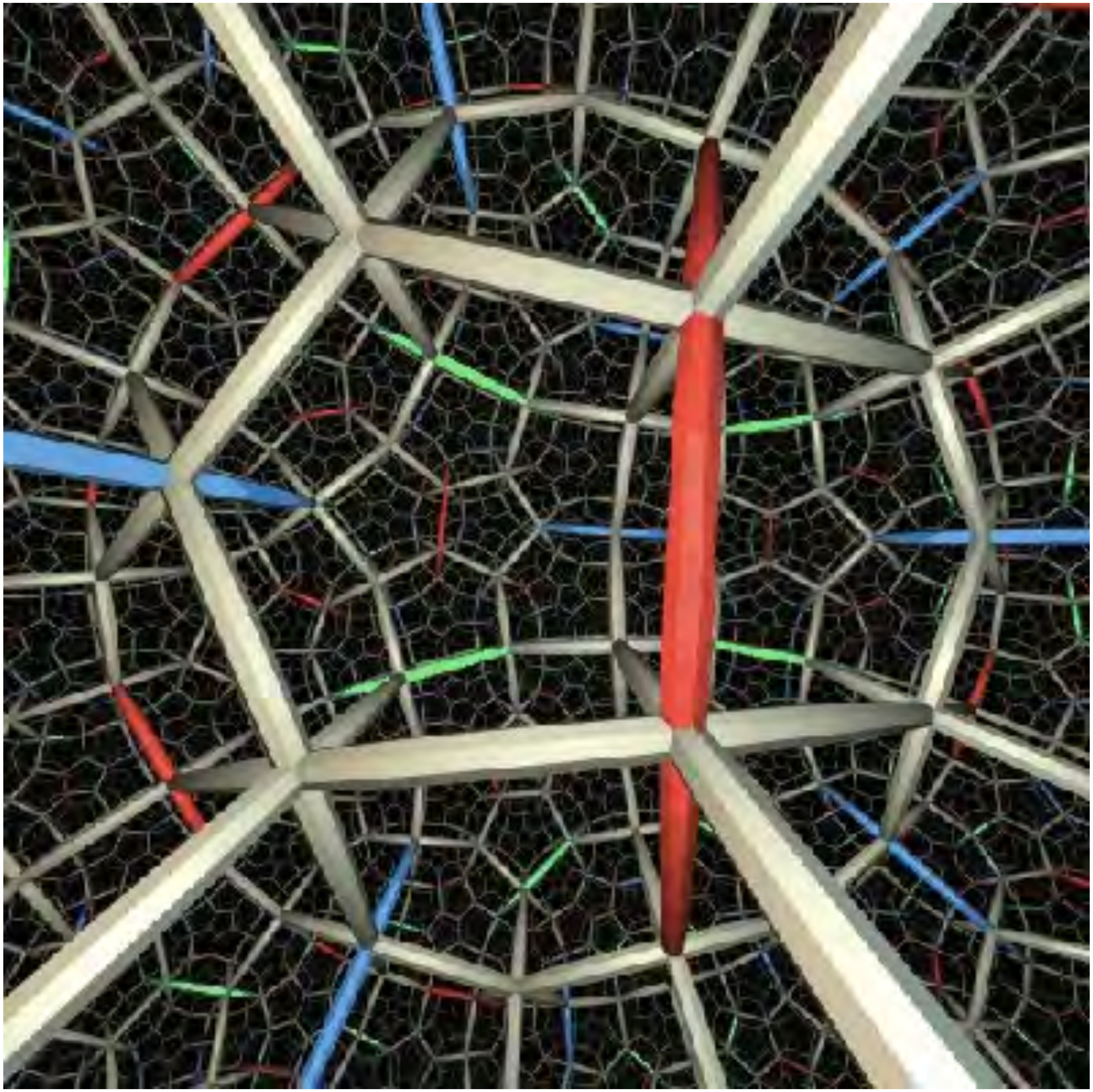


The Poincaré Model Has Correct Angles!

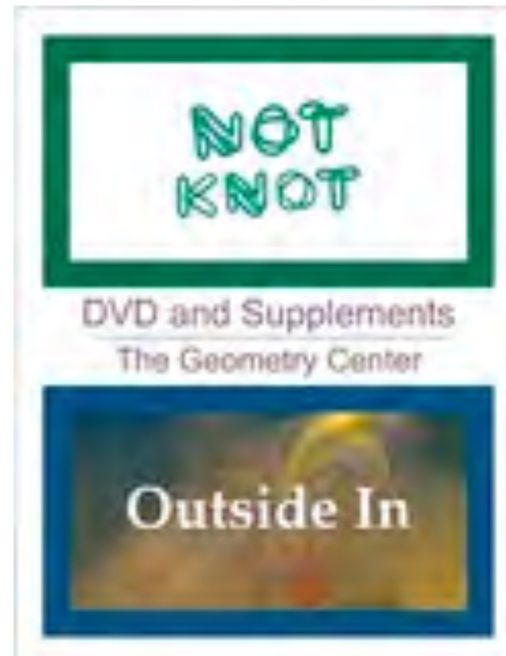




And We Can Go To Higher Dimensions



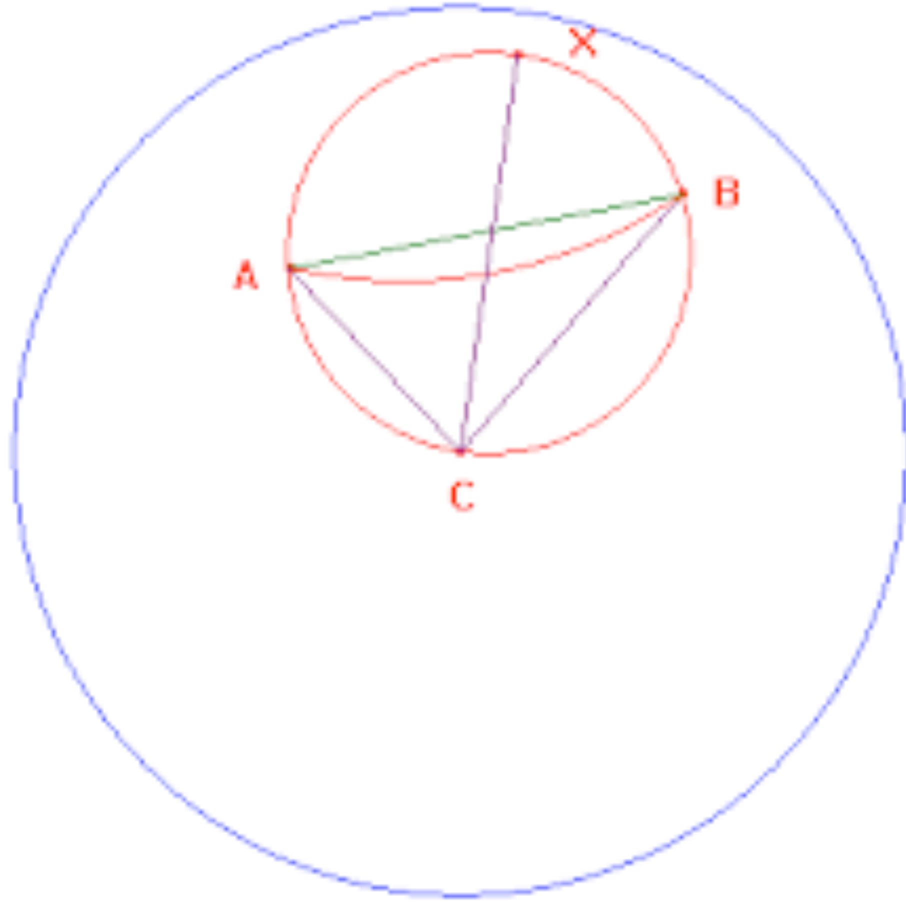
See the film
"Not Knot"



The Poincaré Model Has Correct Angles!

To see that the angle sum in a triangle is < π, just move one vertex to the center of the circle.

Movements in the model preserving *angles* can be done using stereographic projection and rotations of the sphere!



IS MATHEMATICS DISCOVERED OR INVENTED?

Recently a heated debate between realists and relativists in science has erupted. The conflict is between those who see science as a rational description of the world converging on the truth, and those who argue that it is a ***socially constructed account*** of the world, and just one of many possible accounts.

Typically scientists and philosophers of science are realists, arguing that science is approaching a true and accurate description of the real world, whereas social and cultural theorists support a relativist view of science, and argue that all knowledge of the world is socially constructed. What has gone unnoticed in this debate is that there is a parallel and equally fundamental dispute over mathematics.

The absolutist view of mathematics sees it as universal, objective and certain, with mathematical truths being discovered through the intuition of the mathematician and then being established by proof.

Many modern writers on mathematics share this view, including Roger Penrose in "The Emperor's New Mind," and John Barrow in "Pi in the Sky," as indeed do most mathematicians. The absolutists support "discovery" view and argue that mathematical "objects" and knowledge are necessary, perfect and eternal, and remark on the "unreasonable effectiveness" of mathematics in providing the conceptual framework for science. They claim that mathematics must be woven into the very fabric of the world, for since it is a pure endeavour removed from everyday experience how else could it describe so perfectly the patterns found in nature?

Paul Ernest, written in 1996

The Discoverer of Quaternions

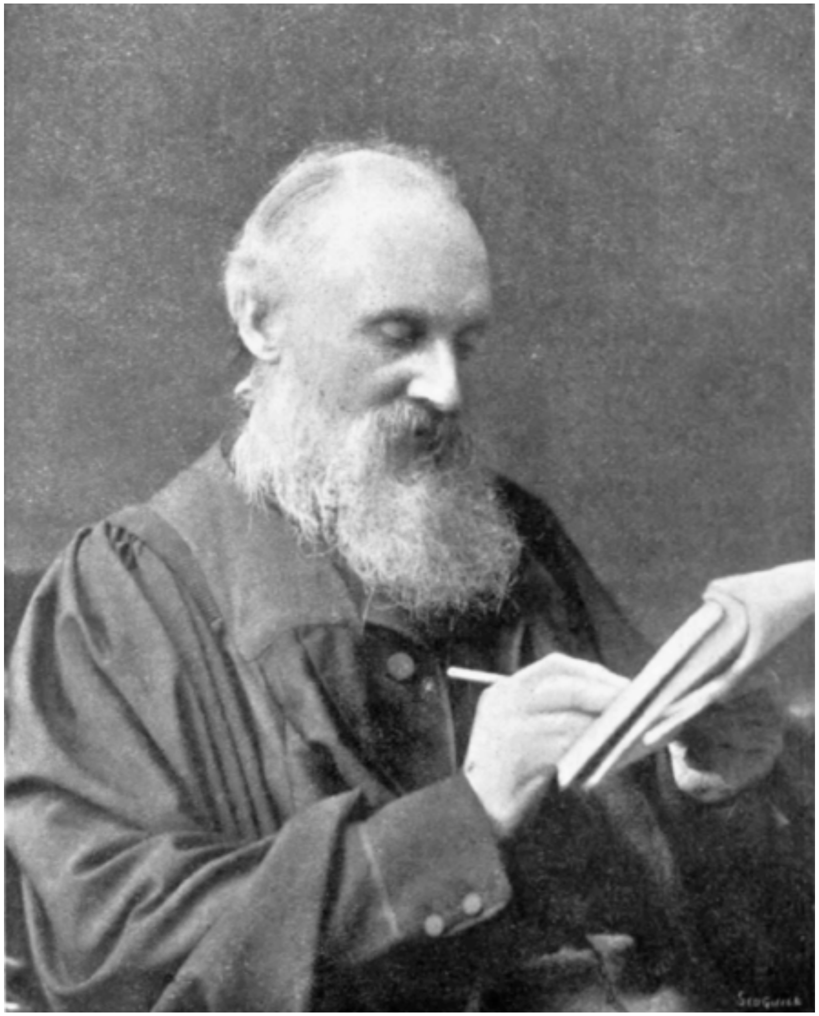
Sir William Rowan Hamilton
(1805 – 1865)

$$i^2 = j^2 = k^2 = i\,j\,k = -1$$

$$q = \alpha + \beta\, i + \gamma\, j + \delta\, k$$

Quaternions Considered Harmful?

"Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clark Maxwell." *(1892)*



Lord Kelvin *(1824 – 1907)*

A Curious Diversion in the History of Physics

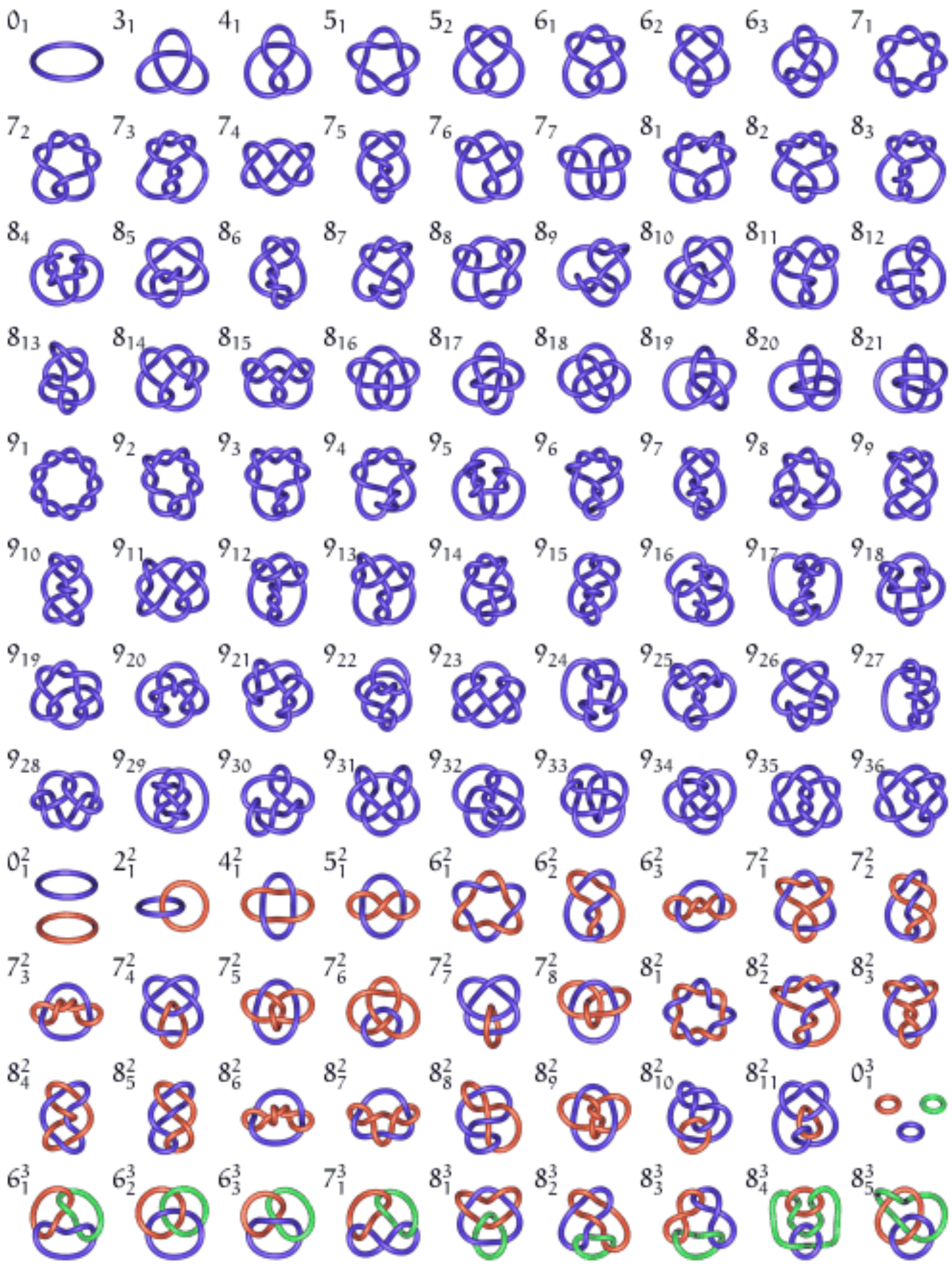
Hermann von Helmholtz (1858) published his seminal paper "*Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*." Soon P. G. Tait published an English translation, "*On integrals of the hydro-dynamical equations which express vortex motion*." Helmholtz established his three **"laws of vortex motion"** in much the same way one finds them in any advanced textbook of **fluid mechanics** today.

William Thomson, later **Lord Kelvin**. was so impressed he put foward the **vortex atom theory** where atoms were to be represented as **vortex motions in the ether**.

This theory predated **quantum theory** by several decades and — because of the scientific standing of its originator — received considerable attention. Many profound insights into vortex dynamics were generated during the pursuit of this theory. One corollary was the first counting of simple knots by **P. G. Tait**, today considered a pioneering effort in **graph theory**, **topology** and **knot theory**.

Ultimately, **Kelvin's** vortex atom was seen to be wrong-headed but the many results in vortex dynamics that it precipitated have stood the test of time. **Kelvin** himself originated the notion of **circulation** and proved that in an **inviscid fluid** circulation around a material contour would be conserved. This profound result – singled out by **Einstein** as one of the most significant results of **Kelvin's** work – provided an early link between fluid dynamics and topology.

Lord Kelvin's "Periodic Table"



Peter Guthrie Tait
(1831 — 1901)

What are Quaternions, Really?

Answer: Quaternions are quotients of 3D vectors!

U/V = U′/V′ iff the triangles OUV and OU′V′ lie in the same plane and are similar and similarly oriented.

Defining the algebra:

$$U/W + V/W = (U + V)/W$$

$$(U/W)\,(W/V) = U/V$$

If you stay in one plane, you get the complex numbers, Hence, there are a continuum number of copies of the complex numbers inside the quaternions.

Discoverers of Geometric Algebra

Hermann Grassmann
(1809 – 1877)

William Kingdon Clifford
(1845 – 1879)

What are Clifford Algebras?

Clifford algebras are **associative algebra**s over the real numbers **freely generated** from a given vector space with an **inner product** satisfying for all vectors:

$$\mathbf{U}\,\mathbf{V} + \mathbf{V}\,\mathbf{U} = 2\,\langle \mathbf{U}, \mathbf{V} \rangle$$

They can be thought of as one of the possible generalizations of the **complex numbers** and **quaternions**.

The theory of Clifford algebras is intimately connected with the theory of **quadratic forms** and **orthogonal transformations**. And they have important applications in a variety of fields including **geometry**, **computer graphics** and **theoretical physics**.

Do You Know Your ABCs?

The ***Fermat-Catalan Conjecture*** combines ideas of Fermat's Last Theorem and the Catalan Conjecture and states that the equation

$$a^m + b^n = c^k$$

has only ***finitely many solutions*** (a, b, c, m, n, k) with distinct triplets of values (a^m, b^n, c^k), and where the a, b, c are positive coprime integers and the m, n, k are positive integers satisfying

$$1/k + 1/m + 1/n < 1.$$

Beal's Conjecture is true if and only if all Fermat-Catalan solutions use **2** as an exponent for some variable.

And as of 2015, only ***ten solutions*** are known:

$1^m+2^3 = 3^2$ $2^5+7^2 = 3^4$ $13^2+7^3 = 2^9$

$2^7+17^3 = 71^2$ $3^5+11^4 = 122^2$ $33^8+1549034^2 = 15613^3$

$1414^3+2213459^2 = 65^7$ $9262^3+15312283^2 = 113^7$

$17^7+76271^3 = 21063928^2$ $43^8+96222^3 = 30042907^2$

Algebra Is Smarter Than You Are!

Jean-Joël Delorme discovered that

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2,$$

provided that $a^2 + b^2 = c^2$, and we let

$$x = (a^2 - b^2)\, c^4, \quad y = 2\, a^2\, b\, c^3, \quad z = 2\, a\, b^2\, c^3, \quad \text{and}$$
$$t = 2\, a\, b\, (a^4 + b^4).$$

Verification in ***Mathematica***:

```
x^4+y^4+z^4+t^4-(x^2+y^2+z^2-t^2)^2//Factor
```

$$8\, a^2\, b^2\, c^6\, (a^2 + b^2 - c^2)\, (4\, a^{10}\, b^2 + 8\, a^6\, b^6 + 4\, a^2\, b^{10} + a^{10}\, c^2 + a^8\, b^2\, c^2 + 2\, a^6\, b^4\, c^2 + 2\, a^4\, b^6\, c^2 + a^2\, b^8\, c^2 + b^{10}\, c^2 + a^8\, c^4 + 2\, a^4\, b^4\, c^4 + b^8\, c^4 + a^6\, c^6 - a^4\, b^2\, c^6 - a^2\, b^4\, c^6 + b^6\, c^6)$$

Computer algebra and computer-assisted theorem proving is changing Mathematics.

What If Ramanujan Had Mathematica?

Ramanujan did his calculations by hand -- with chalk on slate, or later pencil on paper. Today with ***Mathematica*** and the Wolfram Language we have immensely more powerful tools with which to do experiments and make discoveries in mathematics (not to mention the computational universe in general).

It's fun to imagine what Ramanujan would have done with these modern tools. I rather think he would have been quite an adventurer -- going out into the mathematical universe and finding all sorts of strange and wonderful things, then using his intuition and aesthetic sense to see what fits together and what to study further.

Ramanujan unquestionably had remarkable skills. But I think the first step to following in his footsteps is just to be adventurous: not to stay in the comfort of well-established mathematical theories, but instead to go out into the wider mathematical universe and start finding -- ***experimentally*** -- what's true.

It's taken the better part of a century for many of Ramanujan's discoveries to be fitted into a broader and more abstract context. But one of the great inspirations that Ramanujan gives us is that it's possible with the right sense to make great progress even before the broader context has been understood. And I for one hope that many more people will take advantage of the tools we have today to follow Ramanujan's lead and make great discoveries in experimental mathematics -- whether they announce them in unexpected letters or not.

Stephen Wolfram — 2016 Blog

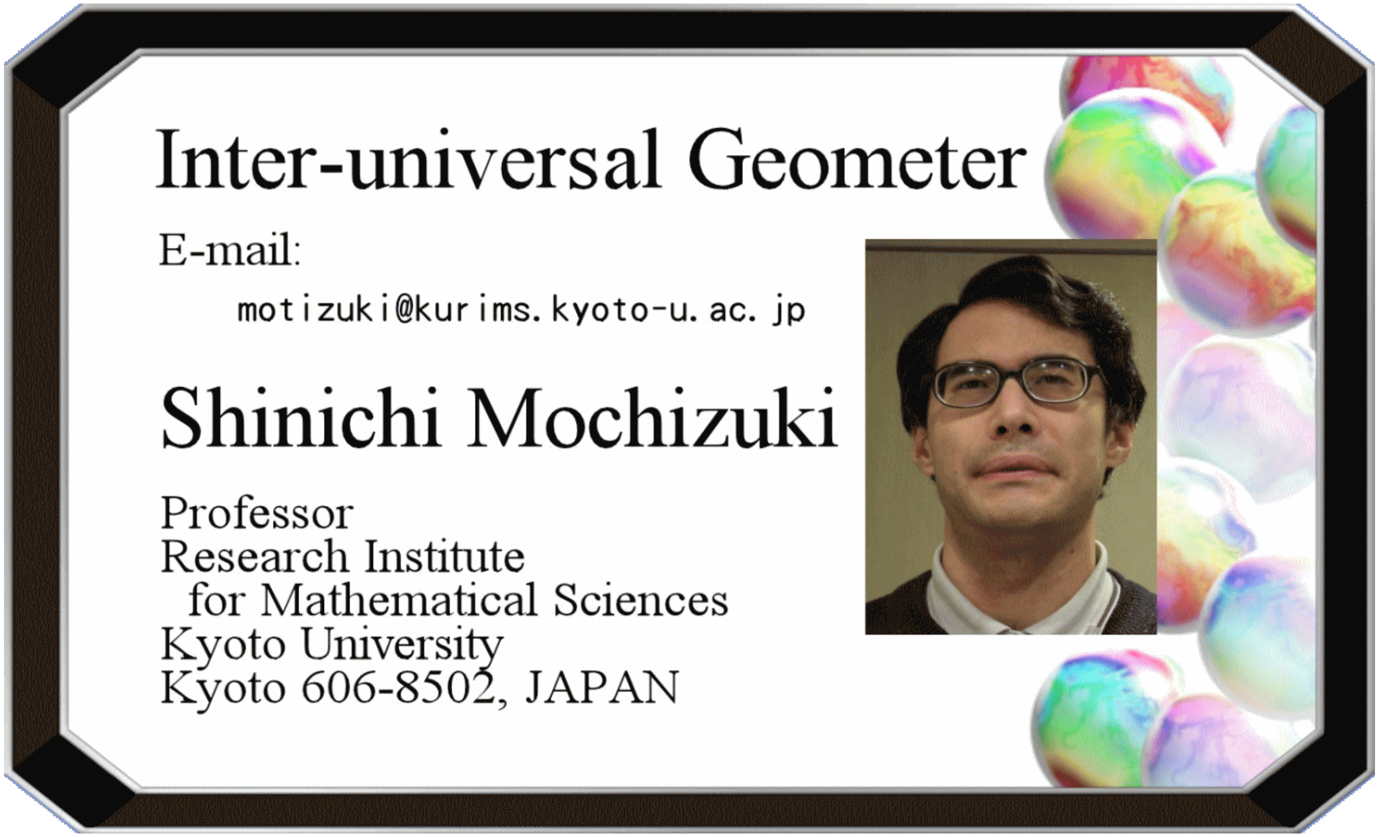
When does a Proof become a PROOF?

A proof becomes a proof only after the social act of "accepting it as a proof." This is as true for mathematics as it is for physics, linguistics, or biology.

The evolution of commonly accepted criteria for an argument's being a proof is an almost untouched theme in the history of science. In any case, the ideal for what constitutes a mathematical demonstration of a "nonobvious truth" has remained unchanged since the time of Euclid: we must arrive at such a truth from "obvious" hypotheses, or assertions that have already been proved, by means of a series of explicitly described, "obviously valid" elementary deductions.

Yuri I. Manin with Boris Zilber. "A Course in Mathematical Logic for Mathematicians." 2nd ed., p. 58, Graduate Texts in Mathematics, vol. 53, Springer, 2010, xii + 384 pp.

An Indigestible Proof?



http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html

Princeton University Ph.D. under Gerd Faltings in 1992 at age 23. Known for a proposed proof of **abc conjecture**, and the proof of the Grothendieck conjecture on **anabelian geometry**.

Homage to Two Pioneers!

Alexander Grothendieck (1928 – 2014)
Professeur
Institut des Hautes Études Scientifiques (1958 – 1970)

The leading figure in the creation of modern algebraic geometry with research extending the scope of the field and adding elements of commutative algebra, homological algebra, sheaf theory and category theory to its foundations.

Robert Phelan Langlands (1936 –)
Hermann Weyl Professor Emeritus
Institute for Advanced Study, Princeton (1972 – 2007)

The Langlands Program is a vast web of far-reaching and influential conjectures and results that relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles.

Does Mathematics need a Foundation?

Answer:

Does Mathematics need a Foundation?

Answer: Yes & No!

One reason the working mathematician can ignore the question of need of ***foundational axioms*** is that the mathematics of the 99% group … can easily be formalized in ZFC and, in fact, in ***much weaker systems***.

Indeed, research in recent years in ***predicative mathematics*** and in the ***reverse mathematics program*** shows that the bulk of it can be formalized in subsystems of analysis hardly stronger than Π^1_1-CA, and moreover the ***scientifically applicable part*** can be formalized in systems conservative over PA and even much weaker systems.

So, foundationally, everyday mathematics rests in principle on ***unexceptionable grounds***.

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Princeton University Press: Science Essentials, 2015, xxii + 464 pp.

A Suggestion

- In view of the big ***progress in Mathematics*** since 2000
- In view of the many recent ***books and writings***
- In view of the advances in ***computer-based proofs***
- In view of the renewed interest in ***constructive reasoning***

It should be a very good time to have
seminars and discussion groups
on proofs and logic
to chart future directions.

Thank You

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